Exam #1

SOLUTIONS

1. { 15 points } **Find the total area of the region between the curves** $y = x^2$ **and** $y = \sin\left(\frac{\pi}{2}x\right)$ **for** $-1 \le x \le 1$.

$$\begin{aligned}
A &= \int_{-1}^{0} x^2 - \sin\left(\frac{\pi}{2}x\right)\ dx \quad + \quad \int_{0}^{1} \sin\left(\frac{\pi}{2}x\right) - x^2\ dx \\
&= \frac{x^3}{3}\Big|_{-1}^{0} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{-1}^{0} \quad - \quad \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{0}^{1} - \frac{x^3}{3}\Big|_{0}^{1} \\
&= \frac{1}{3}(0-(-1)) + \frac{2}{\pi}(1-0) - \frac{2}{\pi}(0-1) - \frac{1}{3}(1-0) \quad = \quad \frac{4}{\pi}
\end{aligned}$$

2. { 15 points } **Find the total area of the region enclosed by the curves** $y = 7 - x$ **and** $y = \sqrt{25-4x}$.

To find the intersection points of the curves we solve

$$7 - x = \sqrt{25-4x}.$$

Squaring both sides of the equation we receive

$$49 - 14x + x^2 = 25 - 4x \qquad \Leftrightarrow \qquad x^2 - 10x + 24 = 0$$

which have solutions $x = 4$ **and** $x = 6$.
In addition, we see that $\sqrt{25-4x} \ge 7 - x$. **Thus,**

$$A = \int_4^6 \sqrt{25-4x} - (7-x)\ dx = \frac{1}{-4}\int_4^6 \sqrt{25-4x}\ d(25-4x) \ + \ \int_4^6 x - 7\ dx$$

$$\begin{aligned}
&= -\frac{1}{4}\left(\frac{2}{3}\right)(25-4x)^{\frac{3}{2}}\Big|_4^6 \quad + \quad \left(\frac{x^2}{2} - 7x\right)\Big|_4^6 \\
&= -\frac{1}{6}\left((25-24)^{\frac{3}{2}} - (25-16)^{\frac{3}{2}}\right) + \frac{1}{2}\left(6^2 - 4^2\right) - 7(6-4) \\
&= -\frac{1}{6}\left(1^{\frac{3}{2}} - 9^{\frac{3}{2}}\right) + \frac{1}{2}(36-16) - 14 \quad = \quad -\frac{1}{6}(1-27) + 10 - 14 \\
&= \frac{26}{6} - 4 \quad = \quad \frac{13}{3} - \frac{12}{3} \quad = \quad \frac{1}{3}
\end{aligned}$$

3. { 15 points } **Let** R **be the region between the curves** $y = \sqrt{x}$ **and** $y = \frac{2}{\sqrt{x}}$ **for** $1 \le x \le 4$. **Find and evaluate a definite integral that represents the volume of the solid generated by revolving** R **about the** x**-axis.**

Since $\sqrt{x} \le \dfrac{2}{\sqrt{x}}$ **on the interval** $[1, 2]$ **and** $\sqrt{x} \ge \dfrac{2}{\sqrt{x}}$ **on** $[2, 4]$**, we have**

$$V = \pi \int_1^2 \left(\frac{2}{\sqrt{x}}\right)^2 - (\sqrt{x})^2 \; dx \; + \; \pi \int_2^4 (\sqrt{x})^2 - \left(\frac{2}{\sqrt{x}}\right)^2 \; dx$$

$$= \pi \int_1^2 \left(\frac{4}{x} - x\right) \; dx \; + \; \pi \int_2^4 \left(x - \frac{4}{x}\right) \; dx$$

$$= \pi \left(4 \; \ln x \Big|_1^2 - \frac{x^2}{2}\Big|_1^2 \; + \; \frac{x^2}{2}\Big|_2^4 - 4 \; \ln x \Big|_2^4 \right)$$

$$= \pi \left(4 \; \ln 2 - \frac{1}{2}(4-1) + \frac{1}{2}(16-4) - 4 \; \ln 4 + 4 \; \ln 2 \right)$$

$$= \left(4 \; \ln 2 - \frac{3}{2} + 6 - 4 \; \ln \frac{4}{2} \right) \; = \; \frac{9\pi}{2}$$

4. { 15 points } **Consider the region** R **enclosed between** $y = \sqrt{9 - x^2}$ **and** $y = 2 - x$ **for** $0 \le x \le 2$. **Find the volume of the solid generated by revolving** R **about the** y**-axis by integrating with respect to** x.

We use cylindrical shells with heights $\sqrt{9-x^2} - (2-x) \ge 0$ **for** $0 \ge x \ge 2$

$$V = 2\pi \int_0^2 x \left(\sqrt{9 - x^2} - (2 - x) \right) \; dx$$

$$= 2\pi \int_0^2 x\sqrt{9 - x^2} \; dx \; - \; 2\pi \int_0^2 x(2 - x) \; dx$$

$$= 2\pi \left(-\frac{1}{2}\right) \int_0^2 \sqrt{9 - x^2} \; (-2x) \; dx \; + \; 2\pi \int_0^2 x^2 - 2x \; dx$$

Change $u = 9 - x^2$ **in the first integral. Then** $du = -2x\,dx$ **and the limits change as follows:** $x = 0 \to u = 9$ **and** $x = 2 \to u = 9 - 2^2 = 5$

$$V = -\pi \int_9^5 \sqrt{u}\, du + 2\pi \left(\frac{x^3}{3} - x^2\right)\Big|_0^2 \; = \; -\pi \, \frac{2}{3} u^{\frac{3}{2}} \Big|_9^5 + 2\pi \left(\frac{1}{3}(2^3 - 0^3) - (2^2 - 0^2) \right)$$

$$= -\frac{2\pi}{3} \left(5\sqrt{5} - 9\sqrt{9} \right) + 2\pi \left(\frac{8}{3} - 4 \right) \; = \; -\frac{10\sqrt{5}\pi}{3} + 18\pi - \frac{8\pi}{3} \; = \; \frac{46\pi}{3} - \frac{10\sqrt{5}\pi}{3}$$

5. { 15 points } **Find the exact arc length of the parametric curve without eliminating the parameter**

$$x = \cos t + t\sin t\ , \qquad y = \sin t - t\cos t \qquad (0 \le t \le \pi)$$

$$\begin{aligned} x' &= -\sin t + \sin t + t\cos t \;=\; t\cos t \\ y' &= \cos t - \cos t + t\sin t \;=\; t\sin t \end{aligned}$$

$$L \;=\; \int_0^{\pi} \sqrt{(x')^2 + (y')^2}\ dt \;=\; \int_0^{\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\ dt$$

$$= \int_0^{\pi} \sqrt{t^2}\ dt \;=\; \int_0^{\pi} |t|\ dt \;=\; \int_0^{\pi} t\ dt \;=\; \frac{t^2}{2}\Big|_0^{\pi} \;=\; \frac{\pi^2}{2}$$

6. { 15 points } **Let C be the curve $x - y^2 = 0$ between $x = 0$ and $x = 2$. Find the area of the surface generated by revolving C about the x-axis by integrating with respect to x.**

$$y^2 = x \quad \Rightarrow \quad y(x) = x^{\frac{1}{2}} \quad \Rightarrow \quad y'(x) = \frac{1}{2}\, x^{-\frac{1}{2}}$$

Thus,

$$A \;=\; 2\pi \int_0^{2} y\ \sqrt{1 + (y')^2}\ dx \;=\; 2\pi \int_0^{2} \sqrt{x}\ \sqrt{1 + \frac{1}{4x}}\ dx$$

$$= \;2\pi \int_0^{2} \sqrt{x + \frac{1}{4}}\ dx \;=\; 2\pi\, \frac{2}{3} \left(x + \frac{1}{4}\right)^{\frac{3}{2}} \Bigg|_0^{2}$$

$$= \frac{4\pi}{3} \left(\left(\frac{9}{4}\right)^{\frac{3}{2}} - \left(\frac{1}{4}\right)^{\frac{3}{2}} \right) \;=\; \frac{4\pi}{3} \left(\frac{27}{8} - \frac{1}{8} \right) \;=\; \frac{13\pi}{3}$$

7. { 15 points } **Find the average value of** $f(x) = \frac{1}{e^{-x}+e^{x}}$ **over the interval** $\left[-\frac{\ln 3}{2}, 0\right]$.

$$f_{\text{ave}} = \frac{1}{0-\left(-\frac{\ln 3}{2}\right)} \int_{-\frac{\ln 3}{2}}^{0} \frac{1}{e^{-x}+e^{x}}\, dx = \frac{2}{\ln 3} \int_{\ln\left(3^{-\frac{1}{2}}\right)}^{0} \frac{e^x}{e^x\left(e^{-x}+e^{x}\right)}\, dx$$

$$= \frac{2}{\ln 3} \int_{\ln\left(\frac{1}{\sqrt{3}}\right)}^{0} \frac{e^x}{1+e^{2x}}\, dx$$

change of variables: $u = e^x \quad \Rightarrow \quad du = e^x\, dx$ **with limits** $x = \ln\left(\frac{1}{\sqrt{3}}\right) \;\rightarrow\; u = e^{\ln\left(\frac{1}{\sqrt{3}}\right)} = \frac{1}{\sqrt{3}}$ **and** $x = 0 \;\rightarrow\; u = e^0 = 1$

$$f_{\text{ave}} = \frac{2}{\ln 3} \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1+u^2}\, du = \frac{2}{\ln 3} \tan^{-1} u \,\Big|_{\frac{1}{\sqrt{3}}}^{1} = \frac{2}{\ln 3}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6\ln 3}$$

8. { 15 points } **A boat is anchored so that the anchor is** $100\ ft$ **below the surface of the water. In the water, the anchor weights** $1200\ lb$ **and the chain weights** $40\ lb/ft$. **How much work is required to raise the anchor to the surface?**

When the anchor is $x\ ft$ **below the surface the total weight is** $1200 + 40x\ lb$ **and therefore the force is** $F(x) = (1200+40x)\, g$ **, where** g **is the gravity constant. Thus,**

$$W = \int_0^{100} F(x)\, dx = \int_0^{100} (1200+40x) g\, dx = \left(20x^2 + 1200x\right) g \Big|_0^{100}$$

$$= (200,000 + 120,000)\, g = 320,000\, g$$