Solutions for Quiz #5

September 18, 2003

1. { 1 point for each correct answer } **For each of the following statements determine whether it is true or false.** *It is not necessary to justify your answer.*

(A) The square of any integer has the form $4k$ or $4k+1$ for some integer k. **true**

For even integers $n = 2p$ we have $n^2 = 4p^2 = 4k$. For odd integers $n = 2p+1$ we have $n^2 = 4p^2 + 4p + 1 = 4(p^2 + p) + 1 = 4k + 1$.

(B) For all real numbers x, $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$. **false**

Counterexample: for $x = 1.5$ we have $\lfloor x^2 \rfloor = \lfloor 2.25 \rfloor = 2 \neq 1 = \lfloor 1.5 \rfloor^2 = 1^2$.

(C) For all real numbers x, $\lceil x+1 \rceil = \lceil x \rceil + 1$. **true**

Let $n = \lceil x \rceil$. Then by the definition $n-1 < x \leq n$. Hence $n < x+1 \leq n+1$ and therefore $\lceil x+1 \rceil = n+1 = \lceil x \rceil + 1$.

(D) There is no least positive rational number. **true**

Suppose not. Let $x = \frac{a}{b} > 0$ be the least positive rational number, where a and $b \neq 0$ are integers. Then $\frac{x}{2} = \frac{a}{2b} > 0$ is rational and $0 < \frac{x}{2} < x$, which is a contradiction.

(E) For all integers a and n, if $a \mid n^2$ then $a \mid n$. **false**

Counterexample: for $a = 4$ and $n = 2$ we have $a = 4 \mid 4 = n^2$ but $a = 4 \nmid 2 = n$.

2. { 5 points } **Prove the statement.** Recall that the symbol $\nmid$ means "*does not divide*".

For all integers a, b, and c, if $a \nmid bc$ then $a \nmid b$.

Suppose not.
Then there exist integers a, b, and c such that $a \nmid bc$ and $a \mid b$.
From the definition of $a \mid b$ there exists an integer k such that $b = ka$. Then $bc = kac = (kc)a$.
But kc is an integer. Therefore $a \mid bc$, which is a contradiction.

The statement can be proved by contraposition. The contrapositive statement is:

For all integers a, b, and c , if $a \mid b$ then $a \mid bc$.

To prove of the last statement we assume that a, b, and c are arbitrary integers and $a \mid b$. Then there exists an integer k such that $b = ka$. Thus $bc = kac = (kc)a$.
Since kc is an integer, we have $a \mid bc$.

Bonus Problem. { 5 points } **Prove the statement.**

$4\sqrt{2} - 9$ is irrational.

Suppose not, i.e. the number is rational.
Then there exist integers a and $b \neq 0$ such that $4\sqrt{2} - 9 = \frac{a}{b}$.
Hence $4\sqrt{2} = \frac{a}{b} + 9 = \frac{a+9b}{b}$ and therefore $\sqrt{2} = \frac{a+9b}{4b}$.
But $a + 9b$ and $4b \neq 0$ are integers.
Thus, $\sqrt{2}$ is a rational number, which is a contradiction.