Solutions for Quiz #4

September 11, 2003

1. { 1 point for each correct answer } **For each of the following statements determine whether it is true or false.** *It is not necessary to justify your answer.*

(A) The difference of any two odd integers is even. **true**
Let $n = 2p + 1$ and $m = 2q + 1$ be two arbitrary odd integers.
Then $n - m = 2p - 2q = 2(p - q)$ is even.

(B) For all integers m, if $m > 2$ then $m^2 - 4$ is composite. **false**
Counterexample: for $m = 3$ we have $m^2 - 4 = 5$ which is a prime number.

(C) For all nonnegative real numbers a and b, $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. **false**
Counterexample: for $a = 1$ and $b = 1$ we have $\sqrt{a+b} = \sqrt{2} \neq \sqrt{a} + \sqrt{b} = 2$

(D) The product of any two even integers is a multiple of 4. **true**
Let $n = 2p$ and $m = 2q$ be two arbitrary even integers.
Then $n.m = 2p.2q = 4(pq)$ is a multiple of 4.

(E) For all integers n, $n(6n + 3)$ is divisible by 3. **true**
We have $n(6n + 3) = 3(2n^2 + 3n)$ which by the definition is divisible by 3.

(F) For all integers a and b, if $a \mid 10b$ then $a \mid 10$ or $a \mid b$. **false**
Counterexample: for $a = 4$ and $b = 2$ we have $a = 4 \mid 20 = 10b$ but $a = 4 \nmid 10$ and $a = 4 \nmid 2 = b$.

2. { 4 points } **Prove the statement.**

The square of any odd integer is odd.

Let $n = 2p + 1$ **be an arbitrary odd integer. Then**

$$n^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

is odd by the definition.

Bonus Problem. { 5 points } **Prove the statement.**

Given any two rational numbers r and s with $r < s$, there is a rational number x such that $r < x < s$.

In order to prove the statement, we have to find a rational number x **which satisfies** $r < x < s$**. For the number** $x = \frac{r+s}{2}$ **we have**

$$r = \frac{r+r}{2} < \frac{r+s}{2} < \frac{s+s}{2} = s\ .$$

It remains to prove that x **is rational. Let** $r = \frac{a}{b}$ **and** $s = \frac{c}{d}$ **where** a, $b \neq 0$, c, **and** $d \neq 0$ **are integers. Then**

$$x = \frac{r+s}{2} = r = \frac{a}{2b} + s = \frac{c}{2d} = \frac{ad+cb}{2bd}\ ,$$

where $ad + cb$ **and** $2bd \neq 0$ **are integers. Therefore** x **is rational.**