Solutions for Quiz #10

October 30, 2003

1. { 2 points } **Find the first four terms of the recursively defined sequence**
$t_k = t_{k-1} + 2t_{k-2}$, for all integers $k \geq 2$; $t_0 = -1$, $t_1 = 1$.

$$t_2 = t_1 + 2t_0 = 1 - 2 = -1 , \qquad t_3 = t_2 + 2t_1 = -1 + 2 = 1 ,$$
$$t_4 = t_3 + 2t_2 = 1 - 2 = -1 , \qquad t_5 = t_4 + 2t_3 = -1 + 2 = 1 .$$

2. { 4 points } **Given the sequence** $C_n = \frac{1}{n+1}\binom{2n}{n}$, **for each integer** $n \geq 1$ **,**

show that it satisfies the recurrence relation $C_k = \frac{4k-2}{k+1} C_{k-1}$, **for all integers** $k \geq 2$ **.**

$$\frac{\frac{4k-2}{k+1} C_{k-1}}{C_k} = \frac{4k-2}{k+1} \cdot \frac{\frac{1}{k}\binom{2k-2}{k-1}}{\frac{1}{k+1}\binom{2k}{k}} = \frac{4k-2}{k} \cdot \frac{\binom{2k-2}{k-1}}{\binom{2k}{k}}$$
$$= \frac{4k-2}{k} \cdot \frac{(2k-2)!}{(k-1)!\,(k-1)!} \cdot \frac{k!\,k!}{(2k)!} = \frac{4k-2}{k} \cdot \frac{(2k-2)!}{(2k)!} \cdot \frac{k!\,k!}{(k-1)!\,(k-1)!}$$
$$= \frac{4k-2}{k} \cdot \frac{1}{(2k)(2k-1)} \cdot \frac{(k)(k)}{1} = 1.$$

This proves $\frac{\frac{4k-2}{k+1} C_{k-1}}{C_k} = 1$, and therefore $\frac{4k-2}{k+1} C_{k-1} = C_k$.

3. { 4 points } **Use iteration to guess an explicit formula for the recursively defined sequence**

$v_k = v_{k-1} + 2^k$, for all integers $k \geq 1$; $v_0 = 1$.

We have $v_1 = v_0 + 2 = 1 + 2$, $v_2 = v_1 + 2^2 = (1+2) + 2^2 = 1 + 2 + 2^2$,
$v_3 = v_2 + 2^3 = (1 + 2 + 2^2) + 2^3$, $v_4 = v_3 + 2^4 = (1 + 2 + 2^2 + 2^3) + 2^4$,
so we can guess

$$v_n = \sum_{i=0}^{n} 2^i = \frac{1 - 2^{n+1}}{1 - 2} = 2^{n+1} - 1 .$$

Bonus. { 4 points } **Prove by induction the formula you received in problem 3.**

Let first check the formula $v_n = 2^{n+1} - 1$ for $n = 0$. So $v_0 = 2^1 - 1 = 1$.
Assume that the formula is valid for $n = k - 1 \geq 0$, i.e. $v_{k-1} = 2^k - 1$.
Then

$$v_k = v_{k-1} + 2^k = 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1 ,$$

and therefore the formula is valid for $n = k$. The induction gives that the formula is valid for all integer $n \geq 0$.