Quiz #13

SOLUTIONS

1. { 10 points } **Give a combinatorial proof that if n is a positive integer, then**

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Let us count the ways to select a pair (A, w) consisting of a subset $A \subseteq X$ of the set X of n elements together with one element $w \in A$ of this subset.

First, we choose a subset A of k elements in $\binom{n}{k}$ ways and then an element w of this set in k ways. That gives $\binom{n}{k}k$ ways for a fixed k between 0 and n. Adding all these numbers for $k = 0, 1, ..., n$ gives the sum on the left.

On the other hand, the particular element w of the subset A can be chosen at the beginning in n different ways. To select the other elements of the subset A we simply choose a subset $C \subseteq (X - \{w\})$ of the remaining $n-1$ elements which can be done in 2^{n-1} ways. That makes a total of $n2^{n-1}$.