

Quiz #10

SOLUTIONS

1. { 10 points } Given that $2^{k-1} < n \leq 2^k$, prove (using mathematical induction) that the **merge sort** algorithm applied to the set of n numbers uses at most

$f(k) =$ _____ comparisons of the elements of the set.

Fill an appropriate function $f(k) = O(k2^k)$ for which you can prove the estimate.

We shall prove the estimate for the function $f(k) = (k - 1)2^k + 1$ which is the best possible.

For $k = 1$ we have $n = 2$ and need only one comparison between the two elements of the set. Since $f(1) = 1$ this proves the estimate for $k = 1$.

Assume that the estimate holds for k . Then, for $k + 1$ we have that the set of at most 2^{k+1} elements can be divided into two subsets of at most 2^k elements each. From the induction hypothesis each of these subset can be sorted using $f(k) = (k - 1)2^k + 1$ comparisons. In the merging procedure we select and put aside one element after each comparison until one of the subsets becomes empty (which means that the other one will still have at least one element). Therefore, to merge the subsets we need another at most $2^{k+1} - 1$ comparisons. That makes the total number of comparisons at most

$$2f(k) + 2^{k+1} - 1 = 2(k - 1)2^k + 2 + 2^{k+1} - 1 = (k - 1 + 1)2^{k+1} + 1 = f(k + 1) .$$

This completes the induction argument.