SOLUTIONS for Exam # 3

1. { 15 points } **Analyze the signs of the derivatives of the function**

$$f(x) = x^3 - 9x^2 + 24x$$

to find the following:

(a) the interval(s) on which f is increasing *Answer:* $(-\infty, 2] \cup [4, +\infty)$

(b) the interval(s) on which f is decreasing *Answer:* $[2, 4]$

(c) the open interval(s) on which f is concave up *Answer:* $(3, +\infty)$

(d) the interval(s) on which f is concave down *Answer:* $(-\infty, 3)$

(e) the x-coordinates of all inflection points *Answer:* $x = 3$

Calculations: **The first derivative is**

$$f'(x) = 3x^2 - 18x + 24 = 3(x^3 - 6x + 8) = 3(x-2)(x-4)$$

and therefore $f'(x) \geq 0$, **if** $x \leq 2$ **or** $x \geq 4$, **and also** $f'(x) \leq 0$ **for** $2 \leq x \leq 4$. **The second derivative is** $f'' = 6x - 18 = 6(x-3)$ **which gives that** $f''(x) > 0$ **for** $x > 3$, $f''(x) < 0$ **for** $x < 3$, **and** $x = 3$ **is the only inflection point.**

2. { 20 points } **Find the absolute maximum and minimum values of the function**

$$f(x) = x^2(x-2)^{\frac{2}{3}} \qquad \textbf{on the interval} \qquad [-1, 3] \ ,$$

and state where those values occur.

The first derivative of the function is

$$f'(x) = 2x(x-2)^{\frac{2}{3}} + x^2\left(\frac{2}{3}\right)(x-2)^{-\frac{1}{3}} = \left(\frac{2}{3}\cdot 3\right)x(x-2)^{1-\frac{1}{3}} + \frac{2}{3}x^2(x-2)^{-\frac{1}{3}}$$

$$= \frac{2}{3}x(x-2)^{-\frac{1}{3}}\Big(3(x-2) + x\Big) = \frac{2}{3}x(x-2)^{-\frac{1}{3}}(4x-6) = \frac{8}{3}x(x-2)^{-\frac{1}{3}}\left(x - \frac{3}{2}\right).$$

Thus, f' **is not defined for** $x = 2$ **and** $f'(x) = 0$ **for** $x = 0$ **and** $x = \frac{3}{2}$. **All three critical points are in the interval** $[-1, 3]$. **We evaluate** $f(x)$ **at them and at the ends of the interval**

$$f(-1) = 1^2(-3)^{\frac{2}{3}} = \sqrt[3]{9}\ ; \quad f(0) = 0\ ; \quad f\left(\frac{3}{2}\right) = \frac{9}{4}\left(-\frac{1}{2}\right)^{\frac{2}{3}} = \frac{9}{4\sqrt[3]{4}}\ ;$$

$$f(2) = 0\ ; \quad f(3) = 3^2\ 1^{\frac{2}{3}} = 9$$

and receive that the absolute maximum of f **is** 9 **at** $x = 3$ **and the absolute minimum is** 0 **at** $x = 0$ **and at** $x = 2$.

3. { 15 points } **A sheet of cardboard $6\ in$ square is used to make an open box by cutting squares of equal size x from the corners and folding up the sides. What should be x to obtain a box with largest possible volume?**

The base of the box is a square with length of the side $6-2x$, and the hight of the box is x. Thus, the volume of the box is $V(x)=(6-2x)^2x$. Since the dimensions of the box cannot be negative, we have that $6-2x\geq 0$ and $x\geq 0$ which gives that $x\in[0,3]$. To find the critical points we examine the derivative

$$V'(x)=2(6-2x)(-2)x+(6-2x)^2=(6-2x)(-4+6-2x)=4(3-x)(1-x)\,.$$

So, the critical points are $x=3$ and $x=1$. We evaluate $V(x)$ at 0, 1, and 3 and receive $V(0)=0$, $V(1)=4^2(1)=16$, $V(3)=0$. Hence, the maximal volume is 16 cubic inches at $x=1$.

4. { 15 points } **Determine whether all of the hypotheses of the Mean-Value Theorem are satisfied on the given interval. If not, state which hypotheses fail; if so, find all values of c in that interval that satisfy the conclusion of the theorem.**

$$f(x)=\frac{x+3}{x-1}\quad;\qquad[2,3]$$

The function $f(x)$ is continuous for all real $x\neq 1$ and in particular in the interval $[2,3]$. The derivative

$$f'(x)=\frac{(x-1)-(x+3)}{(x-1)^2}=\frac{-4}{(x-1)^2}$$

is defined also for all real $x\neq 1$ and therefore for all x in the interval $(2,3)$. Thus, both hypotheses of the Mean-Value Theorem are satisfied and therefore there exists a point $c\in(2,3)$ such that

$$f'(c)=\frac{-4}{(c-1)^2}\quad=\quad\frac{f(3)-f(2)}{3-2}=\frac{\frac{3+3}{3-1}-\frac{2+3}{2-1}}{1}=3-5=-2$$

From the above equation we receive that $\frac{-4}{(c-1)^2}=-2$ and therefore $(c-1)^2=2$. Solving the equation gives $c-1=\pm\sqrt{2}$ and thus $c=1\pm\sqrt{2}$. Since $1-\sqrt{2}$ is not in $(2,3)$, we have that the only solution is $c=1+\sqrt{2}$.

5. { 15 points } **Evaluate the integral.**

$$\int \left(\frac{2+x^3}{x} - \cos x + \frac{3}{\sqrt{1-x^2}} \right)\ dx$$

$$= \int \left(\frac{2}{x} + x^2 \right)\ dx\ - \int \cos x\ dx\ + 3 \int \frac{1}{\sqrt{1-x^2}}\ dx$$

$$= 2\ln|x| + \frac{x^3}{3} - \sin x + 3\sin^{-1} x + C$$

6. { 15 points } **Evaluate the integral**

$$\int \sqrt{x^{-\frac{2}{3}} \left(x^{\frac{2}{3}} + 1 \right)}\ \ dx$$

by making the substitution $u = 1 + x^{\frac{2}{3}}$ **.**

We have that $du = \left(1 + x^{\frac{2}{3}}\right)' dx = \ \frac{2}{3} x^{-\frac{1}{3}}\ dx = \frac{2}{3}\ \sqrt{x^{-\frac{2}{3}}}\ dx$**. Thus,**

$$\int \sqrt{x^{-\frac{2}{3}} \left(x^{\frac{2}{3}} + 1 \right)}\ dx\ =\ \frac{3}{2} \int \sqrt{u}\ du\ =\ \frac{3}{2}\ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C\ =\ \left(1 + x^{\frac{2}{3}}\right)^{\frac{3}{2}} + C$$

7. { 15 points } **Expand the sum for** $n = 3$ **and find its value.**

$$\sum_{k=1}^{n} \frac{k(n-k)}{n}$$

Express the above sum in closed form for arbitrary n**.**

$$\sum_{k=1}^{3} \frac{k(3-k)}{3} = \frac{(1)(3-1)}{3} + \frac{(2)(3-2)}{3} + \frac{(3)(3-3)}{3} = \frac{2}{3} + \frac{2}{3} + \frac{0}{3} = \frac{4}{3}$$

$$\sum_{k=1}^{n} \frac{k(n-k)}{n}\ =\ \sum_{k=1}^{n} \frac{kn-k^2}{n}\ =\ \sum_{k=1}^{n} k\ -\ \frac{1}{n} \sum_{k=1}^{n} k^2$$

$$= \frac{n(n+1)}{2} - \frac{1}{n}\ \frac{n(n+1)}{2}\ \frac{2n+1}{3}\ =\ \frac{3n^2+3n}{6} - \frac{(n+1)(2n+1)}{6}$$

$$= \frac{3n^2+3n}{6} - \frac{2n^2+3n+1}{6}\ =\ \frac{n^2-1}{6}$$

For $n = 3$ **the above formula gives** $\frac{9-1}{6} = \frac{8}{6} = \frac{4}{3}$ **which confirms the first calculation.**

8. { 15 points } **Evaluate the integral using the Fundamental Theorem of Calculus.**

$$\int_0^1 \left(e^x + \frac{2}{1+x^2} - \sqrt{4x} \right)\, dx$$

$$= \int_0^1 e^x dx + 2\int \frac{dx}{1+x^2} - \sqrt{4}\int x^{\frac{1}{2}}\, dx = e^x\Big|_0^1 + 2\tan^{-1} x\Big|_0^1 - 2\left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^1$$

$$= e^1 - e^0 + 2\left(\tan^{-1} 1 - \tan^{-1} 0\right) - \frac{4}{3}\left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right) = e - 1 + 2\left(\frac{\pi}{4} - 0\right) - \frac{4}{3}$$

$$= e + \frac{\pi}{2} - \frac{7}{3}$$

First Bonus Problem. { 15 points } **Use the Fundamental Theorem of Calculus to find the derivative.**

$$\frac{d}{dx}\int_{\sqrt{x}}^{x^2} (\ln t)^2\, dt$$

Let $F(x) = \int (\ln x)^2\, dx$ **. Then from the Fundamental Theorem of Calculus**

$$\int_{\sqrt{x}}^{x^2} (\ln t)^2\, dt = F(x^2) - F(\sqrt{x})$$

and using that $F'(x) = (\ln x)^2$ **we receive**

$$\frac{d}{dx}\int_{\sqrt{x}}^{x^2} (\ln t)^2\, dt = F'(x^2)(2x) - F'(\sqrt{x})\frac{1}{2\sqrt{x}}$$

$$= \left(\ln x^2\right)^2(2x) - \left(\ln\sqrt{x}\right)^2\frac{1}{2\sqrt{x}} = (\ln x)^2\left(8x - \frac{1}{8\sqrt{x}}\right)$$

Second Bonus Problem. { 15 points } **Evaluate the integral using an appropriate substitution.**

$$\int_0^1 \frac{\sin\left(\frac{\pi}{x+1}\right)}{(x+1)^2}\, dx$$

Let $u = \frac{\pi}{x+1}$ **. Then** $du = \frac{-\pi}{(x+1)^2}\, dx$ **.**

Using that $x = 0 \mapsto u = \pi$ **and** $x = 1 \mapsto u = \frac{\pi}{2}$ **we receive**

$$\int_0^1 \frac{\sin\left(\frac{\pi}{x+1}\right)}{(x+1)^2}\, dx = -\frac{1}{\pi}\int_{\pi}^{\frac{\pi}{2}} \sin u\, du = \frac{1}{\pi}\cos u\Big|_{\pi}^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\cos\frac{\pi}{2} - \cos\pi\right) = \frac{1}{\pi}\Big(0 - (-1)\Big) = \frac{1}{\pi}$$