Solutions for Exam #2

November 6, 2003

1. { 10 points: +2 points for a right answer, −1 point for a wrong one, 0 points if none or both of the boxes are checked. *In case the total is negative, 0 points will be given for this problem.* }
For each of the following statements determine whether it is true or false.

It is not necessary to justify your answers.

(A) $\{2\} \subseteq \{1, \{2\}, \{3\}\}$. **false**

$\{2\}$ is an element, not a subset.

(B) $3 \in \{1, \{2\}, \{3\}\}$. **false**

$\{3\}$ is an element, but 3 is not.

(C) For all sets A, B, and C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$. **true**

Let $x \in A \cap C$. Then $x \in A$ and $x \in C$. Since from $A \subseteq B$ and $x \in A$ it follows that $x \in B$, we receive $x \in B \cap C$.

(D) For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$. **false**

counterexample: $A = C = \{1\}$ and $B = \{2\}$.

(E) For all sets A and B, if $B \subseteq A^c$ then $A \cap B = \emptyset$. **true**

Let $x \in B$. Then from $B \subseteq A^c$ it follows that $x \notin A$. Hence, the sets A and B have no common elements.

2. { 10 points } **Suppose** $A = \{1, 3, 5\}$ **and** $B = \{2, 3\}$. **Find each of the following.**

$A \cup B = \{1, 2, 3, 5\}$.

$A - B = \{1, 5\}$.

$\mathcal{P}(A) = \big\{\ \emptyset, \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}, \{1, 3, 5\}\ \big\}$.

$\mathcal{P}(A \cap B) = \mathcal{P}(\{3\}) = \big\{\ \emptyset, \{3\}\ \big\}$.

$A \times B = \big\{\ (1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\ \big\}$.

3. { 10 points } **A combination lock requires four selections of numbers, each from 0 to 9.**

(A) **How many different combinations are possible?**

$$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\ 000\ .$$

(B) **Suppose the lock is constructed in such a way that no number can be used twice. How many different combinations are possible?**

$$10 \cdot 9 \cdot 8 \cdot 7 = 5\ 040 \qquad \left(\; = P(10,4) = \frac{10!}{6!} \right)\ .$$

4. { 10 points } **Suppose that four computer boards in a production run of sixty are defective. A sample of five is to be selected to be checked for defects.**

The answers can be given as arithmetic expressions, if you do not use a calculator.

(A) How many different samples can be chosen?

$$n_A = \binom{60}{5} = \frac{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 5\ 461\ 512\ .$$

(B) How many samples will contain at least one defective board?

$60-4 = 56$ boards are not defective, so $\binom{56}{5}$ samples will not have a defective board. Thus, the answer is

$$n_B = \binom{60}{5} - \binom{56}{5} = \frac{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56 - 56 \cdot 55 \cdot 54 \cdot 53 \cdot 52}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 1\ 641\ 696\ .$$

(C) What is the probability that a randomly chosen sample of five will contain at least one defective board?

$$P = \frac{n_B}{n_A} = \frac{60 \cdot 59 \cdot 58 \cdot 57 - 55 \cdot 54 \cdot 53 \cdot 52}{60 \cdot 59 \cdot 58 \cdot 57} \approx 0.30059\ .$$

5. { 10 points } **Use the Binomial Theorem to show that for all integers** $n \geq 0$**,**

$$4^n = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + ... + 3^n\binom{n}{n} \ .$$

$$\binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + ... + 3^n\binom{n}{n} = \sum_{i=0}^{n} 3^i\binom{n}{i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} 1^{n-i}3^i = (1+3)^n = 4^n \ .$$

6. { 10 points }

How many functions are there from a set with four elements to a set with ten elements?

$$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10\ 000 \ .$$

How many of these functions are one-to-one?

$$10 \cdot 9 \cdot 8 \cdot 7 = 5\ 040 \ .$$

How many of these functions are onto?

$$0$$

Every element from the second set should be assigned as a value of the function. This is impossible since there are ten elements for only four values.

7. { 10 points } **Determine whether or not the function** $f(x) = \frac{2x+1}{x-1}$ **, defined for all real numbers** $x \neq 1$ **, is one-to-one and justify your answer.**

The function is one-to-one. To prove this, we have to show that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. We shall now prove the validity of an equivalent statement, namely its contrapositive: if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Let x_1 and x_2 be arbitrary real numbers different from 1, and such that $f(x_1) = f(x_2)$. Then $\frac{2x_1+1}{x_1-1} = \frac{2x_2+1}{x_2-1}$, which gives $(2x_1+1)(x_2-1) = (2x_2+1)(x_1-1)$. After the multiplication we receive $2x_1x_2 - 2x_1 + x_2 - 1 = 2x_1x_2 - 2x_2 + x_1 - 1$. A simplification gives $3x_2 = 3x_1$, and therefore $x_1 = x_2$.

8. { 10 points } **Show that the sequence** $a_n = 2^n - 1$ **for** $n = 0, 1, 2, ...$ **satisfies the recurrence relation**

$$a_k = 3a_{k-1} - 2a_{k-2}\ , \qquad \text{for all integers } \ k \geq 2\ .$$

We have $a_{k-1} = 2^{k-1} - 1$ and $a_{k-2} = 2^{k-2} - 1$. Thus

$$3a_{k-1} - 2a_{k-2} = 3\left(2^{k-1} - 1\right) - 2\left(2^{k-2} - 1\right) = 3 \cdot 2^{k-1} - 3 - 2 \cdot 2^{k-2} + 2$$
$$= 3 \cdot 2^{k-1} - 2^{k-1} - 1 = 2 \cdot 2^{k-1} - 1 = 2^k - 1 = a_k\ .$$

9. { 10 points } **Show that** $\left(\frac{2}{7}(n+1)(n^2-7)\right)^3$ **is** $\mathcal{O}(n^9)$ **.**

$$\left(\frac{2}{7}(n+1)(n^2-7)\right)^3 = \left(\mathcal{O}(n)\ \mathcal{O}(n^2)\right)^3 = \left(\mathcal{O}(n^3)\right)^3 = \mathcal{O}(n^9)\ .$$

10. { 10 points } **Prove the statement assuming that** n **is an integer variable that takes positive integer values.**

$$1 + 2 + 2^2 + 2^3 + 2^4 + ... + 2^n \quad \text{is} \quad \mathcal{O}(2^n)\ .$$

$$1 + 2 + 2^2 + 2^3 + 2^4 + ... + 2^n = \sum_{i=0}^{n} 2^i = \frac{1 - 2^{n+1}}{1 - 2} = 2^{n+1} - 1 \leq 2 \cdot 2^n\ .$$

First Bonus Problem. { 10 points } **Use iteration to guess an explicit formula for the recursively defined sequence. Prove by induction the formula you received.**

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}}\ , \qquad \text{for all integers } \ k \geq 1\ ; \qquad b_0 = 1\ .$$

Let us evaluate the first five terms of the sequence

$$b_0 = 1\ , \qquad b_1 = \frac{b_0}{1 + b_0} = \frac{1}{1+1} = \frac{1}{2}\ , \qquad b_2 = \frac{b_1}{1 + b_1} = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}\ ,$$

$$b_3 = \frac{b_2}{1 + b_2} = \frac{\frac{1}{3}}{1 + \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4}\ , \qquad b_4 = \frac{b_3}{1 + b_3} = \frac{\frac{1}{4}}{1 + \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{5}{4}} = \frac{1}{5}\ .$$

guess: $\qquad b_n = \dfrac{1}{n+1}\ .$

Proof. For $k = 0$, $b_0 = \frac{1}{0+1} = 1$ and the formula is valid.
Let $k \geq 1$ and let assume that the formula is valid for $n = k - 1$, i.e. $b_{k-1} = \frac{1}{(k-1)+1} = \frac{1}{k}$. Then from the recurrence relation we have

$$b_k = \frac{b_{k-1}}{1 + b_{k-1}} = \frac{\frac{1}{k}}{1 + \frac{1}{k}} = \frac{\frac{1}{k}}{\frac{k+1}{k}} = \frac{1}{k+1} \quad .$$

Thus, by induction, $b_n = \frac{1}{n+1}$ for all integers $n \geq 0$.

Second Bonus Problem. { 10 points } **An ordinary deck of cards contains 52 cards divided into four suits ($\clubsuit$, $\diamondsuit$, $\heartsuit$, $\spadesuit$). Each suit contains 13 cards of following denominations:** $2, 3, 4, 5, 6, 7, 8, 9, 10, J, D, K, A$. **A sample of 5 cards is selected randomly from a deck of cards.**

Give your answers as arithmetic expressions. It is not necessary to calculate the result.

(A) How many different samples are possible?

$$n_A = \binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 52 \cdot 51 \cdot 5 \cdot 49 \cdot 4 = 2\ 598\ 960.$$

(B) Determine the number of different samples such that all of the cards have the same suit?

For each of the four suits we have $\binom{13}{5}$ possible samples. Thus,

$$n_B = 4 \cdot \binom{13}{5} = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{120} = 4 \cdot 13 \cdot 11 \cdot 9 = 5\ 148.$$

(C) Determine the number of different samples such that three of the cards have one denomination, and the other two have another one?

We can choose the first denomination in 13 different ways, and then the second one in 12 ways. Then, we can choose three cards out of four for the first denomination in $\binom{4}{3} = 4$ different ways, and two out of four for the second in $\binom{4}{2} = 6$ ways. Thus,

$$n_C = 13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} = 13 \cdot 12 \cdot 4 \cdot 6 = 3\ 744.$$

(D) Determine the number of different samples such that exactly three of the cards have the same denomination, and the denominations of the other two are different?

We can choose a denomination for the first three cards in 13 different ways, a denomination for the fourth card in $13 - 1 = 12$ ways, and a denomination for the fifth card in $13 - 2 = 11$ ways. In determining the suits, we have $\binom{4}{3} = 4$ possibilities for the first three cards, $\binom{4}{1} = 4$ possibilities for the fourth card, and $\binom{4}{1} = 4$ possibilities for the fifth card. Thus,

$$n_D = 13 \cdot 12 \cdot 11 \cdot \binom{4}{3} \cdot \binom{4}{1} \cdot \binom{4}{1} = 13 \cdot 12 \cdot 11 \cdot 4 \cdot 4 \cdot 4 = 109\ 824.$$

(E) Compare the probabilities of the situations described in **(B)**, **(C)**, and **(D)**.

The probabilities are $P_B = \dfrac{n_B}{n_A}$, $P_C = \dfrac{n_C}{n_A}$, and $P_D = \dfrac{n_D}{n_A}$.

Since $n_C < n_B < n_D$, we have $P_C < P_B < P_D$.

The numbers are

$$P_B = \frac{5\ 148}{2\ 598\ 960} \approx 0.19808\% \ , \qquad P_C = \frac{3\ 744}{2\ 598\ 960} \approx 0.14406\% \ ,$$

$$P_D = \frac{109\ 824}{2\ 598\ 960} \approx 4.22569\% \ .$$