1. { 10 points } **Construct truth tables for the following statement forms.**

$$\sim p \vee (q \wedge r) \qquad \text{and} \qquad \sim p \vee r \rightarrow q$$

p	q	r	$\sim p$	$q \wedge r$	$\sim p \vee (q \wedge r)$	$\sim p \vee r$	$p \vee r \rightarrow q$
T	T	T	F	T	**T**	T	**T**
T	T	F	F	F	**F**	F	**T**
T	F	T	F	F	**F**	T	**F**
T	F	F	F	F	**F**	F	**T**
F	T	T	T	T	**T**	T	**T**
F	T	F	T	F	**T**	T	**T**
F	F	T	T	F	**T**	T	**F**
F	F	F	T	F	**T**	T	**F**

Check the appropriate box:

☐ The statements are equivalent

✓ **The statements are not equivalent**

2. { 10 points } **Write negations of the following statements.**

(A) Today is Thursday and this test is easy.

Today is not Thursday or this test is not easy.

(B) If Helen lives in Athens, then she lives in Greece.

Helen lives in Athens and she does not live in Greece.

3. { 10 points } **Use the truth table to determine whether the following argument is valid.**

$$\begin{array}{c} r \\ p \vee q \\ p \to r \\ \therefore\ p \to \sim q \end{array}$$

p	q	r	$p \vee q$	$p \to r$		$\sim q$	$p \to \sim q$
T	T	T	T	T	**critical row**	F	**F**
T	T	F	T	F			
T	F	T	T	T	**critical row**	T	**T**
T	F	F	T	F			
F	T	T	T	T	**critical row**	F	**T**
F	T	F	T	T			
F	F	T	F	T			
F	F	F	F	T			

Check the appropriate box:

☐ The argument is valid

✓ **The argument is not valid**

Explain: $p \to \sim q$ is not true on one of the critical rows (the first one).

4. { 10 points } **Indicate whether the following argument is valid or invalid.**

All people are mice.

All mice are mortal.

$\therefore$ All people are mortal.

✓ **The argument is valid.** ☐ The argument is not valid.

Support your answer by drawing diagrams:

5. { 10 points } **Rewrite the statements formally using quantifiers and variables, and write a negation for each statement.**

(A) All roses are red.

formally: $\forall x \in \{$ **roses** $\}$, x **is red.**

negation: $\exists x \in \{$ **roses** $\}$, x **is not red.**

or There is a rose which is not red.

(B) For any rational number, there is an integer which is greater than it.

formally: $\forall x \in Q \ , \ \exists n \in Z \ : \ n > x$.

negation: $\exists x \in Q \ : \ \forall n \in Z \ , \ n \leq x$.

6. { 10 points } **Prove the statement.**

For all rational numbers r and s , if $s \neq 0$ then $\frac{r}{s}$ is rational.

Let r and s be arbitrary rational numbers. Then there are integers a, $b \neq 0$, c, and $d \neq 0$ such that $r = \dfrac{a}{b}$ and $s = \dfrac{c}{d}$. In addition, we have that $c \neq 0$, since $s \neq 0$. So, $\dfrac{r}{s} = \dfrac{ad}{bc}$. But ad and bc are integers, and $bc \neq 0$ because of $b \neq 0$ and $c \neq 0$. Therefore $\dfrac{r}{s}$ is rational.

7. { 10 points: +2 points for a right answer, −1 point for a wrong one, 0 points if none or both of the boxes are checked. *In case the total is negative, 0 points will be given for this problem.* }
For each of the following statements determine whether it is true or false.

It is not necessary to justify your answers.

(A) The difference of any two odd integers is odd. false

(B) For all integers a, b, and c, if $a \mid (b + c)$ then $a \mid b$ and $a \mid c$. false

(C) For all integers a, b, and c, if $a \mid b$ then $a \mid bc$. true

(D) There is no greatest negative real number. true

(E) For all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even. true

8. { 10 points } **Prove the statement.**

For all integers n, if n^2 is odd then n is odd.

We shall prove a statement that is equivalent to it. First, we formulate the contrapositive statement:

For all integers n , if n is not odd then n^2 is not odd.

Then, using that if an integer is not odd then it is even, we rephrase the statement as follows:

For all integers n , if n is even then n^2 is even.

To prove the last statement, we let n be arbitrary even integer. Then there exists an integer k such that $n = 2k$. Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ is even, since $2k^2$ is an integer.

9. { 10 points: +2 points for a right answer, −1 point for a wrong one, 0 points if none or both of the boxes are checked. *In case the total is negative, 0 points will be given for this problem.* }
For each of the following statements determine whether it is true or false.

It is not necessary to justify your answers.

(A) For all odd integers n, $\lceil n/2 \rceil = (n+1)/2$. true

(B) For all real numbers x and y, $\lceil xy \rceil = \lfloor x \rfloor \lceil y \rceil$. false

(C) $\sqrt{3}$ is irrational. true

(D) $\sqrt{4}$ is irrational. false

(E) For any integer a, $4 \nmid (a^2 - 2)$. true

10. { 10 points } **Transform each of the following by making the change of variable.**

(A) $j = m + 1$

$$\sum_{m=3}^{8} \left(\frac{1}{m} - \frac{1}{m+1} \right) = \sum_{j=3+1}^{8+1} \left(\frac{1}{j-1} - \frac{1}{j} \right) = \sum_{j=4}^{9} \left(\frac{1}{j-1} - \frac{1}{j} \right).$$

(B) $i = k - 1$

$$\prod_{k=2}^{n+1} \frac{(k-1)^2}{k} = \prod_{i=2-1}^{n+1-1} \frac{(i)^2}{i+1} = \prod_{i=1}^{n} \frac{(i)^2}{i+1}.$$

First Bonus Problem. { 10 points } **Prove the statement by mathematical induction.**

For all integers $n \geq 1$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \ .$$

A solution is to be added soon.

Second Bonus Problem. { 10 points } **Sometimes you do not know whether some statements are true or false. Suppose that** p **and** q **are such statements, and try to get as many points as possible even in the worst case scenario.**

(A) { $+2$ points for a right answer, -1 point for a wrong one, 0 points if none or both of the boxes are checked }

p	☐ true | false ☐
q	☐ true | false ☐
$\sim q$	☐ true | false ☐
$p \vee q$	☐ true | false ☐
$p \wedge q$	☐ true | false ☐
$p \longrightarrow q$	☐ true | false ☐

A solution is to be added soon.

(B) **Explain your strategy in (A).**

A solution is to be added soon.

(C) **Can you get positive number of points in the worst case, if the rules are the following**

{ $+1$ point for a right answer, -1 point for a wrong one, 0 points if none or both of the boxes are checked } ?

Why?

A solution is to be added soon.