High School Math Contest

University of South Carolina

February 4th, 2017

Problem 1. If $(x-y)^2 = 121$ and $(x+y)^2 = 169$, what is xy?

(a) 11 (b) 12 (c) 13 (d) 24 (e) 48

Answer: (b)

Solution: Note that $4xy = (x+y)^2 - (x-y)^2 = 169 - 121 = 48$. $\square$

Problem 2. Suppose the function $g(x) = f(x) + x^3$ is even and $f(-10) = 2017$. What is the value of $f(10)$?

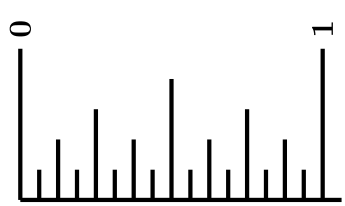
(a) -2017 (b) 17 (c) 1000 (d) 2017 (e) None of the above

Answer: (b)

Solution: Since $g(x)$ is even, $f(x)$ satisfies $f(x) = f(-x) - 2x^3$. Set $x = 10$ to obtain the correct answer. $\square$

Problem 3. A 12 inch ruler has tick marks every sixteenth of an inch: at each inch (including at the beginning and the end) it has a tick mark which is $\frac{5}{10}$ inches long; at each half inch a tick mark $\frac{4}{10}$ inches long; at each quarter inch a tick mark $\frac{3}{10}$ inches long; at each eighth inch a tick mark $\frac{2}{10}$ inches long; at each sixteenth inch a tick mark $\frac{1}{10}$ inches long.

The following diagram illustrates the first portion of the ruler:



What is the total length of all the tick marks on the 12 inch ruler?

(a) $\frac{371}{10}$ (b) $\frac{372}{10}$ (c) $\frac{373}{10}$ (d) $\frac{376}{10}$ (e) $\frac{377}{10}$

Answer: (e)

Solution: There are 13 marks of length $\frac{5}{10}$, 12 of length $\frac{4}{10}$, 24 of length $\frac{3}{10}$, 48 of length $\frac{2}{10}$, and 96 of length $\frac{1}{10}$.

$$13 \cdot \frac{5}{10} + 12 \cdot \frac{4}{10} + 24 \cdot \frac{3}{10} + 48 \cdot \frac{2}{10} + 96 \cdot \frac{1}{10} = \frac{377}{10}. \qquad \square$$

Problem 4. If you know that $f(x)$ is a quadratic polynomial with $f(2) = 8$, $f(3) = 15$, and $f(4) = 26$, what is the sum of the coefficients of f?

(a) 1 (b) 3 (c) 4 (d) 5 (e) 6

Answer: (d)

Solution: Suppose $f(x) = ax^2 + bx + c$. Then we know

$$4a + 2b + c = 8, \quad 9a + 3b + c = 15, \quad 16a + 4b + c = 26.$$

Upon subtracting the first equation from the second, and the second from the third, we obtain $5a + b = 7$ and $7a + b = 11$. We conclude that $a = 2$, $b = -3$, and $c = 6$. $\square$

Problem 5. An equilateral triangle and a regular hexagon have perimeters of the same length. If the area of the triangle is 2 square units, what is the area of the hexagon?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 6

Answer: (c)

Solution: Let a be the side length of the triangle T; then the side length of the hexagon is $a/2$. The hexagon can be cut up into six equilateral triangles, each of side length $a/2$, and each has area $1/4$ that of T. Therefore, the area of the hexagon is $\frac{6}{4}$ times that of T, so 3. $\square$

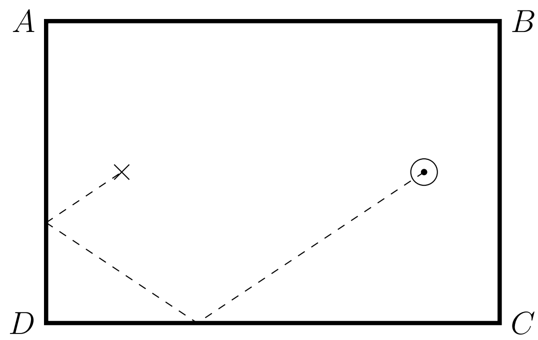
Problem 6. A real number θ satisfies $\cos\theta = \tan\theta$. What is the value of $\dfrac{1}{\sin\theta} + \cos^4\theta$?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

Answer: (e)

Solution: The condition implies that $\cos^2\theta = \sin\theta$. And combining with the identity $\cos^2\theta + \sin^2\theta = 1$, we have $\dfrac{1}{\sin\theta} + \cos^4\theta = \dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta} + \sin^2\theta = (1 + \sin\theta) + (1 - \cos^2\theta) = 2 + \sin\theta - \cos^2\theta = 2.$ $\square$

Problem 7. A white billiard ball is on a 4 foot by 6 foot billiard table, with its center one foot away from a four foot wall, and in the middle of the two longer walls. You aim to reach the position symmetrically opposite.



You hit the white ball so that it travels in a straight line to side $\overline{CD}$, bounces off this side so that the angle of reflection equals the angle of incidence, travels in another straight line to side $\overline{AD}$, bounces off this side so that the angle of reflection again equals the angle of incidence, and again travels in a straight line, now to its final position with its center exactly one foot from the opposite side of the table and in the middle of the two longer sides.

How far in feet does the center of the white ball travel from its initial position to its final position?

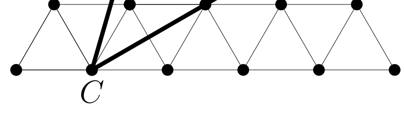
(a) $3 + 2\sqrt{3}$ (b) $4\sqrt{3}$ (c) $2\sqrt{13}$ (d) $2\sqrt{14}$ (e) 7.5

Answer: (c)

Solution: Visualize the target position, off the pool table, one foot below the $\overline{CD}$ wall and two feet to the left of the $\overline{AD}$ wall. Then, a straight line path to this location (jumping over the walls of the pool table) is equivalent to a path to the white ball with two bounces as described. The distance to the target location is $\sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$. $\square$

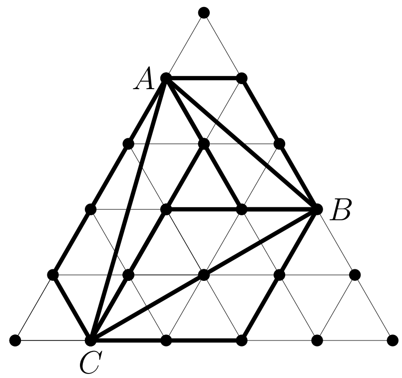
Problem 8. In the figure below, the large equilateral triangle is formed by 25 smaller equilateral triangles, each with an area of 1cm^2. What is the area of triangle $\triangle ABC$ in cm^2?

(a) 10 (b) 10.5 (c) 11 (d) 11.25 (e) 11.5



Answer: (a)

Solution: We divide $\triangle ABC$ into four triangles, and on the outside of $\triangle ABC$ we draw triangles congruent to three of these:



If x is the area of $\triangle ABC$, then since the pairs of congruent triangles cover 18 of the small triangles, we see from the picture that $2(x-1) = 18$. Therefore $x = 10$. $\square$

Problem 9. How many positive integer n satisfy the inequality $1! + 2! + 3! + 4! + n! \leq n^3$?

(a) 1 (b) 2 (c) 4 (d) 5 (e) None of the above

Answer: (a)

Solution: We rewrite the equation as $33 + n! \leq n^3$. Of $n \in \{1, 2, 3, 4, 5\}$ we see that only $n = 4$ yields a solution. No larger n can be a solution, as for $n \geq 5$ the left side grows much more rapidly than the right. (For example, it can be proved that the left side more than doubles at each step with $n \geq 5$, and the right side less than doubles.) $\square$

Problem 10. Suppose a and b are two different real numbers, and the function $f(x) = x^2 + ax + b$ satisfies $f(a) = f(b)$. What is the value of $f(2)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Answer: (d)

Solution: Following the given conditions and the symmetry of the quadratic function with respect to the vertical line at $x = -\frac{a}{2}$, we have $\frac{a+b}{2} = -\frac{a}{2}$. It is then $2a + b = 0$, so $f(2) = 4 + 2a + b = 4$. $\square$

Problem 11. For $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, let $A = (\cos\theta)^{\cos\theta}$, $B = (\sin\theta)^{\cos\theta}$, $C = (\cos\theta)^{\sin\theta}$. Which of the following is true?

(a) $A < B < C$ (b) $A < C < B$ (c) $B < A < C$ (d) $B < C < A$ (e) $C < A < B$

Answer: (e)

Solution: Since $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$, we have $0 < \cos\theta < \sin\theta < 1$. Then $(\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$ and $(\cos\theta)^{\sin\theta} < (\cos\theta)^{\cos\theta}$. $\square$

Problem 12. Find $a \cdot b$, given that

$$a = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{\cdots}}}}$$
$$b = \sqrt{9 - \sqrt{9 - \sqrt{9 - \sqrt{\cdots}}}}$$

(a) $\frac{3}{2}\left(\sqrt{26} - 2\right)$ (b) $\frac{3}{2}\left(\sqrt{37} - 1\right)$ (c) $4\sqrt{5} - 2\sqrt{2}$ (d) $3\sqrt{6} + 1$ (e) None of the above

Answer: (b)

Solution: Notice that $a^2 = 6 + a$; therefore, $a = 3$ (we cannot have $a = -2$). Similarly, note that $b^2 = 9 - b$; thus, $b = (\sqrt{37} - 1)/2$ (we keep only the positive value). From these, it is simple to compute their product. □

Problem 13. How many integers n satisfy $n^4 + 6n < 6n^3 + n^2$?

(a) 3 (b) 4 (c) 5 (d) 6 (e) None of the above

Answer: (b)

Solution: Write the inequality as $n^4 - 6n^3 - n^2 + 6n < 0$, or $n(n-6)(n-1)(n+1) < 0$. Now it is immediate to conclude that $n \in \{2, 3, 4, 5\}$. □

Problem 14. What is the range of the following function?

$$f(x) = \frac{x+1}{x^2+1}$$

(a) $[\frac{1-\sqrt{2}}{4}, \frac{1+\sqrt{2}}{4}]$ (b) $[\frac{1-\sqrt{2}}{3}, \frac{1+\sqrt{2}}{3}]$ (c) $[\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}]$ (d) $[1 - \sqrt{2}, 1 + \sqrt{2}]$ (e) None of the above

Answer: (c)

Solution: Assume $\lambda \in \mathbb{R}$ is in the range of this function. Solve for x in the equation

$$\frac{x+1}{x^2+1} = \lambda$$

to get the equivalent (quadratic) equation $\lambda x^2 - x + (\lambda - 1) = 0$. The discriminant of the solution, $\Delta = 1 - 4\lambda(\lambda - 1)$ indicates precisely what values of λ are acceptable. Solving the corresponding inequality $\Delta \geq 0$ offers the answer. □

Problem 15. Let n be the smallest integer such that the sum of the digits of the sum of the digits of n is 11. What is the sum of the digits of the sum of the digits of $n + 1$?

(a) 3 (b) 4 (c) 7 (d) 8 (e) 12

Answer: (a)

Solution: The sum of the digits of n will be 29 (the smallest number whose digits sum to 11), and so $n = 2999$, and $n + 1 = 3000$. $\square$

Problem 16. Let a, b, c, and d be integers satisfying

$$a \log_{10} 2 + b \log_{10} 3 + c \log_{10} 5 + d \log_{10} 7 = 2017.$$

What is $a + b + c + d$?

(a) 2016 (b) 2017 (c) 5012 (d) 5034 (e) None of the above

Answer: (e)

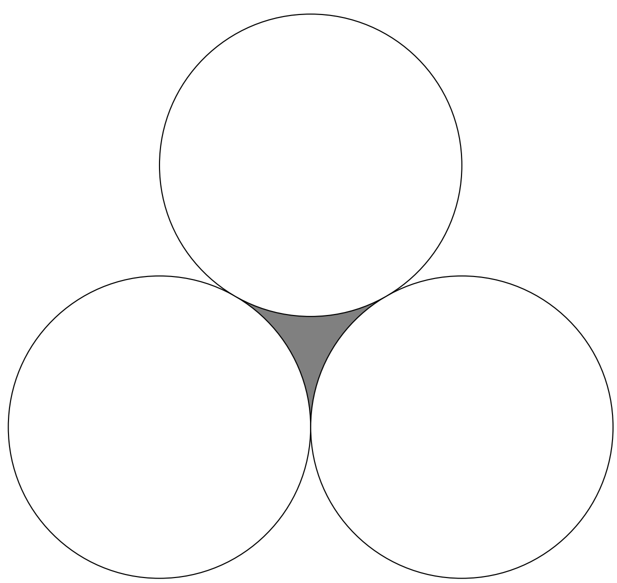
Solution: Raising both sides to the tenth power, we get

$$2^a 3^b 5^c 7^d = 10^{2017},$$

so $a = c = 2017$ and $b = d = 0$. So the answer is 4034, which was not provided. $\square$

Problem 17. Given three circles of radius 2, tangent to each other as shown in the following diagram, what is the area for the shaded region?

(a) $\sqrt{3} - \frac{\pi}{2}$ (b) $4\sqrt{3} - 2\pi$ (c) $\frac{4\sqrt{3}-\pi}{3}$ (d) $2\sqrt{6} - \pi$ (e) None of the above



Answer: (b)

Solution: The shaded area can be obtained by subtracting from the area of an equilateral triangle with side-length 4, three sections of $60°$ of a circle with radius 2. The result follows. $\square$

Problem 18. What is the value of

$$\log_2(27) \cdot \log_3(25) \cdot \log_5(32)$$

to the nearest integer?

(a) 23 (b) 29 (c) 30 (d) 32 (e) 36

Answer: (c)

Solution: Writing everything in terms of natural logarithms, this equals

$$\frac{\ln(27)}{\ln(2)} \cdot \frac{\ln(25)}{\ln(3)} \cdot \frac{\ln(32)}{\ln(5)}.$$

We rearrange this as

$$\frac{\ln(27)}{\ln(3)} \cdot \frac{\ln(25)}{\ln(5)} \cdot \frac{\ln(32)}{\ln(2)} = 3 \cdot 2 \cdot 5 = 30.$$

Note that the instructions to round to the nearest integer are a red herring, as the answer is already an integer. □

Problem 19. In this problem, a and b represent real numbers. Given that the maximum value and the minimum value of the function $f(t) = a \sin t + b \cos t$ differ by exactly 10, what is the largest possible value of $a + b$?

(a) $4\sqrt{2}$ (b) 6 (c) 7 (d) $5\sqrt{2}$ (e) None of the above

Answer: (d)

Solution: For any real numbers a and b, the maximum value of the function $f(t) = a \sin t + b \cos t$ is $\sqrt{a^2 + b^2}$. The minimum value of f is $-\sqrt{a^2 + b^2}$.

Indeed, If θ is an angle such that $\cos \theta = a/\sqrt{a^2 + b^2}$ and $\sin \theta = b/\sqrt{a^2 + b^2}$, then we may write

$$f(t) = \sin(t + \theta)\sqrt{a^2 + b^2}.$$

From the statement of the problem, we deduce $2\sqrt{a^2 + b^2} = 10$. In other words, $a^2 + b^2 = 25$. As $(a + b)^2 = 2(a^2 + b^2) - (a - b)^2 = 50 - (a - b)^2$, we see the maximum value of $a + b$ occurs when $(a + b)^2 = 50$ and $a = b$. This gives $a + b = 5\sqrt{2}$ for the maximum of $a + b$. □

Problem 20. There are 10 sticks in a bag. The length of each stick is an integer (measured in inches). It is not possible to make a triangle out of any three sticks from that bag. What is the shortest length (in inches) the longest stick could possibly be?

(a) 11 (b) 19 (c) 55 (d) 101 (e) 1023

Answer: (c)

Solution: The optimal configuration is when the two shortest sticks are 1 inch each. Then the third must be at least two inches, and will be two inches in an optimal configuration. The fourth must be at least three, the fifth must be at least five $(2+3)$, and so on. We see that the lengths of the sticks are Fibonacci numbers ($8 = 3+5$, $13 = 5+8$, etc.), so the longest has length 55. $\square$

Problem 21. Triangle $\triangle ABC$ has sides $\overline{AB}$ of length 6, $\overline{BC}$ of length 3, and $\overline{AC}$ of length 5. The angle bisector of $\angle C$ intersects $\overline{AB}$ in point D. What is the length of the line segment $\overline{AD}$?

(a) 3 (b) $\frac{13}{4}$ (c) $\frac{7}{2}$ (d) $\frac{15}{4}$ (e) 4

Answer: (d)

Solution: Let $\theta = \angle BCD = \angle ACD$. By the law of sines,

$$\frac{\sin\theta}{BD} = \frac{\sin(\angle BDC)}{3} = \frac{\sin(\angle ADC)}{3} = \frac{5}{3}\cdot\frac{\sin(\angle ADC)}{5} = \frac{5}{3}\cdot\frac{\sin\theta}{AD}.$$

Therefore $BD = \frac{3}{5}AD$, so that $6 = AB = AD + BD = \frac{8}{5}AD$, and $AD = \frac{30}{8} = \frac{15}{4}$. $\square$

Problem 22. For each positive integer n, write $\sigma(n)$ for the sum of the positive divisors of n. As n ranges through all positive integers up to 100, what is the largest value of $\sigma(n)$?

(a) 217 (b) 243 (c) 252 (d) 256 (e) 291

Answer: (c)

Solution: It will be achieved by n which is (a) large and (b) has as many divisors as possible. In particular we want n to be divisible by lots of small numbers (because then n will be divisible by large factors as well).

An obvious candidate is 96, which is divisible by 1, 2, 3, and 4. We have

$$\sigma(96) = 1+2+3+4+6+8+12+16+24+32+48+96 = 252.$$

This is the largest value.

A solution by elimination can be done as follows: Notice that we can write $\sigma(n)$ as the following product:

$$\sigma(n) = \prod_{\substack{p \text{ prime divides } n\\ p^r \text{ divides } n\\ p^{r+1} \text{ does not divide } n}} (1+p+p^2+\cdots+p^r) \tag{1}$$

As the choice 291 is odd and (1) is even when p is odd and r is odd, we are left for this choice with considering n divisible only by a power of 2 or odd prime powers p^r where r is even. For $n \le 100$, this is only possible with the prime powers 2, 4, 8, 16, 32, 64, 9, 81, 25 and 49. As these have (1) equal to 3, 7, 15, 31, 63, 127, 13, 121, 31 and 57, respectively, and these cannot multiply to give $291 = 3\cdot 97$, the choice 291 is not possible. The choice $256 = 2^8$ requires (1) to be a power of 2 for each p^r dividing n as above. In particular, p has to be odd and r has to be odd and one quickly checks $p = 3$ and $r = 3$ to see that (1) is not a power of 2 in this case so that r must be 1 and we are only interested in n which are a product of distinct odd primes p with $1+p$ a power of 2. By considering powers of 2 minus one, we see the only permissible primes p that can divide n are $3= 2^2-1, 7 = 2^3-1$ and $31 = 2^5-1$. But, in order to get $\sigma(n) = 256$, we would need n divisible by both 7 and 31 which does not happen for $n \leq 100$. $\square$

Problem 23. Let x, y and z be real numbers in $[0, 1]$. What is the maximum possible value of $\sqrt{|x-y|} + \sqrt{|y-z|} + \sqrt{|z-x|}$?

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2 (e) $1+\sqrt{2}$

Answer: (e)

Solution: Clearly we should take two of the numbers to be 0 and 1, say $x = 0$ and $y = 1$. Then we maximize the resulting expression by taking z right in the middle, so that $\sqrt{|x-y|} + \sqrt{|y-z|} + \sqrt{|z-x|} = \sqrt{1} + \sqrt{1/2} + \sqrt{1/2} = 1 + \sqrt{2}$. $\square$

Problem 24. The equation $x^3 - 4x^2 - 11x + a$ has three roots x_1, x_2, and x_3. If $x_1 = x_2 + x_3$, what is a?

(a) 24 (b) 28 (c) 30 (d) 32 (e) 36

Answer: (c)

Solution: Because the polynomial in question can be written as $(x-x_1)(x-x_2)(x-x_3)$, we have $x_1 + x_2 + x_3 = 4$. Thus $x_1 = x_2 + x_3 = 2$.

Similarly $x_1x_2 + x_1x_3 + x_2x_3 = -11$, and we have

$$-11 = x_1x_2 + x_1x_3 + x_2x_3 = x_1(x_2 + x_3) + x_2x_3 = 4 + x_2x_3,$$

so that $x_2x_3 = -15$. Thus, $a = -x_1x_2x_3 = 30$. $\square$

Problem 25. Suppose that $t = \sqrt{(x^2-2)^2 + (x-5)^2} + \sqrt{(x^2-3)^2 + x^2}$. As x ranges over all real numbers, what is the minimum value of t?

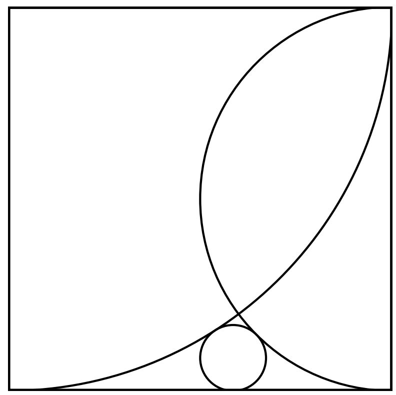
(a) $\sqrt{10}$ (b) $\sqrt{20}$ (c) $\sqrt{26}$ (d) $\sqrt{52}$ (e) There is no minimum

Answer: (c)

Solution: The first square root is the distance between the point (x, x^2) and $(5, 2)$. The second square root is the distance between the point (x, x^2) and $(0, 3)$. As the line segment between $(0, 3)$ and $(5, 2)$ intersects the parabola $y = x^2$, the minimum t is achieved when x is the x-coordinate of this intersection point. The total distance is simply the distance between $(5, 2)$ and $(0, 3)$, or $\sqrt{26}$. $\square$

Problem 26. In a square with side length 1, a quarter circle is drawn with center at the top left corner and passing through the top right and bottom left corners, and a half circle is drawn with the right side of the square as its diameter. (See the accompanying diagram.)

A smaller circle is drawn tangent to the quarter circle, the half circle, and the bottom edge (as shown). Determine its radius.



(a) $\frac{9}{5} - \sqrt{3}$ (b) $\frac{3}{2} - \sqrt{2}$ (c) $3 - 2\sqrt{2}$ (d) $\frac{\sqrt{2}}{2} - \frac{1}{2}$ (e) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

Answer: (b)

Solution: Assume the diagram is oriented so that the bottom left has coordinates $(0, 0)$ and the top right $(1, 1)$. Write (a, x) for the coordinates of the smallest circle; then the radii of the three circles are (from smallest to largest) x, $\frac{1}{2}$, and 1.

The line from $(0, 1)$ to (a, x) has length $1 + x$, as it comprises two radii. Therefore we have $a^2 + (1 - x)^2 = (1 + x)^2$, and hence $a^2 = 4x$. Similarly, considering a line from $(1, \frac{1}{2})$ to (a, x) we conclude that $(1 - a)^2 + (\frac{1}{2} - x)^2 = (\frac{1}{2} + x)^2$, which reduces to $(1 - a)^2 = 2x$.

So $4x = 2(1 - a)^2 = a^2$, so that $a^2 - 4a + 2 = 0$. This quadratic equation has the solutions $a = 2 \pm \sqrt{2}$, of which only $2 - \sqrt{2}$ makes sense geometrically. Therefore $x = \frac{3}{2} - \sqrt{2}$. $\square$

Problem 27. What is the value of $\cos^2 10° + \cos^2 50° - \sin 40° \sin 80°$?

(a) $-\frac{3}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$ (e) $\frac{3}{4}$

Answer: (e)

Solution: We have

$$\begin{aligned}
&\cos^2 10° + \cos^2 50° - \sin 40° \sin 80° \\
&\qquad = (\cos 10° - \cos 50°)^2 + \cos 50° \cos 10° \\
&\qquad = 4 \sin^2 30° \sin^2 20° + \cos 50° \cos 10° \\
&\qquad = \frac{1}{2}(1 - \cos 40°) + \frac{1}{2}(\cos 40° + \cos 60°) \\
&\qquad = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.
\end{aligned}$$

$\square$

Solution: Given the answer choices listed, it is also possible to estimate all but the last answer choice.

The cosine function decays very slowly for small angles; we have $\cos 10° = .984\ldots$ We have that $\cos^2 50°$ is a little bit smaller than $\cos^2 45°$, which is $\frac{1}{2}$. Now we just have to notice that $\sin 40°$ is substantially less than $\frac{3}{4}$, so that $\frac{1}{2}$ and all the smaller answers are not plausible. This leaves $\frac{3}{4}$. $\square$

Problem 28. What is the sum of all integers n that satisfy $\frac{1}{4} < \sin\frac{\pi}{n} < \frac{1}{3}$?

(a) 19 (b) 21 (c) 30 (d) 33 (e) 36

Answer: (d)

Solution: Due to the convexity of $\sin x$ for $x \in (0, \frac{\pi}{6})$, we have $\frac{3}{\pi}x < \sin x < x$. Then $\sin\frac{\pi}{13} < \frac{\pi}{13} < \frac{1}{4}$; $\sin\frac{\pi}{12} > \frac{3}{\pi}\frac{\pi}{12} = \frac{1}{4}$; $\sin\frac{\pi}{10} < \frac{\pi}{10} < \frac{1}{3}$; $\sin\frac{\pi}{9} > \frac{3}{\pi}\frac{\pi}{9} = \frac{1}{3}$. Therefore, we have $\sin\frac{\pi}{13} < \frac{1}{4} < \sin\frac{\pi}{12} < \sin\frac{\pi}{11} < \sin\frac{\pi}{10} < \frac{1}{3} < \sin\frac{\pi}{9}$. So $n = 10, 11, 12$ and their sum is 33. $\square$

Problem 29. A standard die is rolled until a six rolls. Each time a six does not roll, a fair coin is tossed and a running tally of the number of heads minus the number of tails tossed is kept.

For example, if the die rolls are 5, 2, 1, 6 and the tosses are H, H, T, then the running tally is 1, 2, 1.

Find the probability that the absolute value of this running tally never equals 3.

(a) $\frac{289}{414}$ (b) $\frac{7}{10}$ (c) $\frac{81}{112}$ (d) $\frac{3}{4}$ (e) $\frac{5}{6}$

Answer: (a)

Solution: Let $P(x)$ be the probability that the sum will reach ± 3 if it is presently x. If the sum is zero to start, then there is a probability of $5/6 \cdot 1/2$ that it will be 1 after one trial, and $5/6 \cdot 1/2$ that it will be -1. By symmetry, $P(1) = P(-1)$. Expanding probabilities gives

$$\begin{aligned} &P(0) = \tfrac{5}{6} \cdot \tfrac{1}{2}P(1) + \tfrac{5}{6} \cdot \tfrac{1}{2}P(-1) = \tfrac{5}{6}P(1), \\ &P(1) = \tfrac{5}{12}P(0) + \tfrac{5}{12}P(2), \\ &P(2) = \tfrac{5}{12}P(1) + \tfrac{5}{12}. \end{aligned}$$

Solving this system gives $P(0) = 125/414$. The probability that the sum never reaches 3 is thus $1 - P(0) = 289/414$. $\square$

Problem 30. In $\triangle ABC$, if we have $\sin A = 2017 \sin B \sin C$ and $\cos A = 2017 \cos B \cos C$, what is the value of $\tan A$?

(a) 1 (b) $\frac{2018}{2017}$ (c) 2017 (d) 2018 (e) $2017\frac{\sqrt{2}}{2}$

Answer: (d)

Solution: Note that, by hypothesis it must be

$$\begin{aligned} \sin A - \cos A &= 2017 \sin B \sin C - 2017 \cos B \cos C \\ &= -2017 \cos(B + C) \\ &= -2017 \cos(\pi - A) \\ &= 2017 \cos A \end{aligned}$$

The result follows. $\square$