1. How many two digit prime numbers are there in which both digits are prime numbers? (For example, 23 is one of these numbers but 31 is not, since 1 is not a prime number.)

(a) 3  (b) 4  (c) 5  (d) 8  (e) 15

Answer: (b)

Solution: The second digit can only be 3 or 7, so the choice quickly narrows down to 23, 27, 33, 37, 53, 57, 73, and 77. Of these, 27, 33, and 57 are divisible by 3, and 77 by 7, leaving 23, 37, 53, and 73. It is easy to see that none of these is divisible by 2, 3, 5, or 7, and there is no need to look at greater prime divisors since $\sqrt{77} < 11$.

2. You own thirteen pairs of socks, all different, and all of the socks are individually jumbled in a drawer. One morning you rummage through the drawer and continue to pull out socks until you have a matching pair. How many socks must you pull out to guarantee having a matching pair?

(a) 3  (b) 12  (c) 13  (d) 14  (e) 25

Answer: (d)

Solution: You might be unlucky and have the first thirteen socks all different, but then the 14th has to match one of them.

3. A jeweler has a 20 gram ring that is 60% gold and 40% silver. He wants to melt it down and add enough gold to make it 80% gold. How many grams of gold should be added?

(a) 4 grams  (b) 8 grams  (c) 12 grams  (d) 16 grams  (e) 20 grams

Answer: (e)

Solution:

Gold concentration = \( \frac{\text{weight of gold}}{\text{total weight}} \)

\[
0.80 = \frac{20 \cdot 0.60 + x}{20 + x} \\
12 + x = 16 + 0.8x \\
0.2x = 4 \\
x = 20 \text{ grams}
\]
4. Consider the following game. A referee has cards labeled A, B, C, and D, and places them face down in some order. You point to each card in turn, and guess what letter is written on the bottom. You guess each of A, B, C, and D exactly once (otherwise there is no chance of getting them all right!).

You play this game once, and then the referee tells you that you guessed exactly \( n \) of the letters correctly. Which value of \( n \) is not a possible value of \( n \)?

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4

*Answer:* (d)

*Solution:* If the first three are correct, then by process of elimination the fourth has to be correct also. The same reasoning holds no matter when the three correct answers occur.

5. What is the value of \( \sqrt{10 + 4\sqrt{6}} - \sqrt{10 - 4\sqrt{6}} \)?

(a) 1  (b) 4  (c) \( 2\sqrt{6} \)  (d) \( \sqrt{8\sqrt{6}} \)  (e) \( 8\sqrt{6} \)

*Answer:* (b)

*Solution:* Square the expression algebraically, and simplify. Several easy cancellations give us 16, and now we take the square root.

6. The triangle \( \triangle ABC \) has sides of the following lengths: \( AB = 24 \), \( BC = 7 \), and \( AC = 25 \). Let \( M \) be the midpoint of \( AB \). What is the length of \( CM \)? (The figure below is not drawn to scale.)

(a) 1  (b) \( \sqrt{139} \)  (c) 12  (d) \( \sqrt{193} \)  (e) 16

*Answer:* (d)

*Solution:* Since \( 24^2 + 7^2 = 25^2 \), \( \triangle ABC \) is a right triangle. Thus, \( \triangle MBC \) is also a right triangle and \( CM^2 = BM^2 + BC^2 = 12^2 + 7^2 = 193 \).
7. What is the value of \((\log_2 3)(\log_3 4)(\log_4 5) \cdots (\log_{63} 64)\)?

\begin{align*}
(a) \frac{1}{6} & \quad (b) 2 & \quad (c) \frac{5}{2} & \quad (d) 6 & \quad (e) 32 \\
\end{align*}

Answer: (d)

Solution: Using the identity \((\log_a b)(\log_b c) = \log_a c\) repeatedly, we obtain
\[
(\log_2 3)(\log_3 4)(\log_4 5) \cdots (\log_{63} 64) = \log_2 64 = 6.
\]

8. On a test the passing students had an average of 83, while the failing students had an average of 55. If the overall class average was 76, what percent of the class passed?

\begin{align*}
(a) 44\% & \quad (b) 66\% & \quad (c) 68\% & \quad (d) 72\% & \quad (e) 75\% \\
\end{align*}

Answer: (e)

Solution: Let \(p\) = proportion that passed. Then
\[
83p + 55(1 - p) = 76,
\]
so
\[
p = \frac{21}{28} = .75.
\]

9. Jack and Lee walk around a circular track. It takes Jack and Lee respectively 6 and 10 minutes to finish each lap. They start at the same time, at the same point on the track, and walk in the same direction around the track. After how many minutes will they be at the same spot again (not necessarily at the starting point) for the first time after they start walking?

\begin{align*}
(a) 15 & \quad (b) 16 & \quad (c) 30 & \quad (d) 32 & \quad (e) 60 \\
\end{align*}

Answer: (a)

Solution 1: Experimenting with the numbers in turn, dividing 6 into 15 and 10 into 15, we get answers one whole number apart, so they are together again at 15 minutes.

Solution 2: Let \(\text{frac}(x)\) denote the fractional part of \(x\); e.g., \(\text{frac}(3.25) = 0.25\). Let \(x\) be the number of minutes. We solve \(\text{frac}(x/6) = \text{frac}(x/10)\). Thus, \(\text{frac}(x/6 - x/10) = \text{frac}(x/15) = 0\). That is, \(x/15\) is an integer, i.e., \(x\) is a multiple of 15. The smallest such \(x\) is 15.
10. If \( \sin(x) + \cos(x) = \frac{1}{2} \), what is the value of \( \sin^3(x) + \cos^3(x) \)?

(a) \( \frac{1}{8} \)  
(b) \( \frac{5}{16} \)  
(c) \( \frac{3}{8} \)  
(d) \( \frac{5}{8} \)  
(e) \( \frac{11}{16} \)

Answer: (e)

Solution: First, use 
\[
(a^3 + b^3 = (a + b)(a^2 - ab + b^2),
\]
to get
\[
\sin^3(x) + \cos^3(x) = (\sin(x) + \cos(x))((\sin^2(x) - \sin(x)\cos(x) + \cos^2(x)) = \frac{1}{2} \cdot (1 - \sin(x)\cos(x)).
\]
Next, \( \frac{1}{4} = (\sin(x) + \cos(x))^2 = \sin^2(x) + 2\sin(x)\cos(x) + \cos^2(x) = 1 + 2\sin(x)\cos(x) \), thus \( \sin(x)\cos(x) = -\frac{3}{8} \). Combining these, we get
\[
\sin^3(x) + \cos^3(x) = \frac{1}{2} \cdot (1 - (-\frac{3}{8})).
\]

11. The two roots of the quadratic equation \( x^2 - 85x + c = 0 \) are prime numbers. What is the value of \( c \)?

(a) 84  
(b) 166  
(c) 332  
(d) 664  
(e) 1328

Answer: (b)

Solution: Assuming that two prime numbers \( x_1 \) and \( x_2 \) are the solutions, then \( x_1 + x_2 = 85 \).

Since 85 is an odd number, either \( x_1 \) or \( x_2 \) must be an even number. The only even prime number is 2. Therefore, one number must be 2 and the other 83. Hence \( c = 2 \times 83 = 166 \).

12. How many pairs \((x, y)\) of integers satisfy \( x^4 - y^4 = 16 \)?

(a) 0  
(b) 1  
(c) 2  
(d) 4  
(e) infinitely many

Answer: (c)

Solution 1: By Fermat’s Last Theorem, \( x^4 = y^4 + 16 \) can only have \( y \in \{0, 1, -1\} \). Then \( y = 0 \) works and \( y = 1, y = -1 \) do not, so there are two pairs, \((2, 0)\) and \((-2, 0)\).

Solution 2: Observe that the integers \( x^4 \) are rapidly increasing: 0, 1, 16, 81, 256, … . It is apparent that there cannot be any solutions with \( x \) or \( y \) large, as each gap is bigger than 16 when \( y > 2 \).

To complete the solution, write
\[
16 = x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2).
\]

Then we must have \( |x - y| \geq 1 \) and \( |x + y| \geq 1 \), so that \( x^2 + y^2 \leq 16 \), which implies that \( x \) and \( y \) are unequal and between -4 and 4. Moreover, if \((x, y)\) is a solution, so is \((\pm x, \pm y)\), so it is enough to check \( x \) and \( y \) between 0 and 4. This leaves a small number of cases which can easily be checked by hand (or this kind of reasoning may be continued).
13. A circle passes through two adjacent vertices of a square and is tangent to one side of the square. If the side length of the square is 2, what is the radius of the circle?

(a) $\frac{3}{2}$  
(b) $\frac{4}{3}$  
(c) $\frac{5}{4}$  
(d) $\frac{6}{5}$  
(e) None of these

**Answer:** (c)

**Solution:** Let $A$ and $B$ be the vertices through which the circle passes. Let $M$ be the midpoint of the line segment $AB$. The center $C$ of the circle is on the line joining $M$ to the midpoint of the opposite side, which is also on the circle (see diagram below). Let $r$ be the radius of the circle. The length of the line segment joining $M$ to $C$ is $r-(2r-2)=2-r$ as shown below. Then $1^2+(2-r)^2=r^2$ by the Pythagorean Theorem. This gives $1+4-4r+r^2=r^2$, which simplifies to $5-r=0$, so $r=5/4$.

![Diagram of circle and square](image-url)

14. If $x$ and $y$ are positive real numbers, neither of which is equal to 1, what is the smallest non-negative value of $\log_x(y) + \log_y(x)$?

(a) 0  
(b) $\sqrt{2}$  
(c) $\sqrt{\pi}$  
(d) 2  
(e) 10

**Answer:** (d)

**Solution:** Denote $\log_x y$ by $l$. Since $(\log_x y)(\log_y x) = \log_x x = 1$, we need to find the least positive value of $l + \frac{1}{l}$. The last expression is positive only if $l$ is positive. Also, if $l > 0$,

$$l + \frac{1}{l} = 2 + \left(\sqrt{l} - \frac{1}{\sqrt{l}}\right)^2,$$

so the least positive value is 2, and it is achieved when $l = 1$, that is, when $x = y$.

15. What is the value of $\sin\left(\frac{2\pi}{5}\right) + \sin\left(\frac{4\pi}{5}\right) + \sin\left(\frac{6\pi}{5}\right) + \sin\left(\frac{8\pi}{5}\right)$?

(a) 1  
(b) $-1$  
(c) 0  
(d) $\frac{1}{\sqrt{5}}$  
(e) $-\frac{1}{\sqrt{5}}$

**Answer:** (c)

**Solution:** When the points are plotted on the unit circle, the sine is the $y$-coordinate, and they cancel in pairs: $\sin(8\pi/5) = -\sin(2\pi/5)$ and $\sin(6\pi/5) = -\sin(4\pi/5)$.
16. What is the largest integer \( n \) such that \( \frac{n^2 - 38}{n + 1} \) is an integer?

- (a) 36
- (b) 38
- (c) 72
- (d) 76
- (e) None of these

**Answer:** (a)

**Solution:** Doing the long division algebraically, we get a quotient of \( n - 1 \) with a remainder of \(-37\). In order for the division to give an integer, \( n + 1 \) must divide \(-37\) evenly. So \( n + 1 = 37 \), and \( n = 36 \).

17. For a positive integer \( n \), define \( S(n) \) to be the sum of the positive divisors of \( n \). Which of the following is the smallest?

- (a) \( S(2010) \)
- (b) \( S(2011) \)
- (c) \( S(2012) \)
- (d) \( S(2013) \)
- (e) \( S(2014) \)

**Answer:** (b)

**Solution:** 2011 is prime, so that \( S(2011) = 1 + 2011 = 2012 \), the smallest. But this answer can be verified without checking that 2011 is prime. Observe that

\[
S(2010) > 2010 + 1005 > 3000
\]

\[
S(2013) > 2013 + 671 > 2600
\]

and \( S(2012) \) and \( S(2014) \) are also larger than 3000.

Check by hand that 2, 3, 5, 7, and 11 do not divide 2011, and so if 2011 is not prime, then it is a product of two primes \( p \) and \( q \). If \( p \neq q \), then

\[
S(2011) = 2011 + \frac{2011}{p} + \frac{2011}{q} + 1,
\]

and \( p \) and \( q \) are greater than 11, so that this is smaller than 2600. If \( p = q \) then this is smaller still.

18. A class has three girls and three boys. These students line up at random, one after another. What is the probability that no boy is right next to another boy, and no girl is right next to another girl?

- (a) \( \frac{1}{20} \)
- (b) \( \frac{1}{12} \)
- (c) \( \frac{1}{10} \)
- (d) \( \frac{3}{10} \)
- (e) \( \frac{1}{2} \)

**Answer:** (c)

**Solution:** We pick the students in order. The first choice is always okay. There is a probability \( \frac{2}{5} \) that the second choice is okay; for example, if you picked a boy first then there are two boys and three girls left. Similarly the probabilities that the remaining choices are okay is \( \frac{1}{2} \), \( \frac{2}{3} \), \( \frac{1}{2} \), 1. The answer is \( \frac{1}{10} \), the product of all these probabilities.
19. Suppose \( f(x) = ax + b \) and \( a \) and \( b \) are real numbers. We define

\[
f_1(x) = f(x)
\]

and

\[
f_{n+1}(x) = f(f_n(x))
\]

for all positive integers \( n \). If \( f_7(x) = 128x + 381 \), what is the value of \( a + b \)?

(a) 1 \hspace{1cm} (b) 2 \hspace{1cm} (c) 5 \hspace{1cm} (d) 7 \hspace{1cm} (e) 8

Answer: (c)

Solution: From the definition, 

\[
f_n(x) = a^n x + (a^{n-1} + a^{n-2} + \ldots + a + 1) b = a^n x + \frac{a^n - 1}{a - 1} \cdot b
\]

From \( f_7(x) = 128x + 381 \), we get \( a^7 = 128 \), and \( \frac{a^7 - 1}{a - 1} \cdot b = 381 \), therefore, \( a = 2 \), \( b = 3 \), \( a + b = 5 \).

20. A bag contains 11 candy bars: three cost 50 cents each, four cost $1 each and four cost $2 each. How many ways can 3 candy bars be selected from the 11 candy bars so that the total cost is more than $4?

(a) 8 \hspace{1cm} (b) 28 \hspace{1cm} (c) 46 \hspace{1cm} (d) 66 \hspace{1cm} (e) 70

Answer: (c)

Solution: The ways of choosing 3 candy bars with a total cost over $4 include: choose 3 out of 4 (2 dollars each); choose 2 out of 4 (2 dollars each) and 1 from the other 7. So the total number of ways is \( C_3^4 + (7 \times C_2^4) = 46 \).

Incidentally, the total number ways of choosing 3 candy bars out of 11 is \( C_3^{11} = 165 \). So the probability of them costing more than $4 if they are randomly chosen is

\[
\frac{C_3^4 + (7 \times C_2^4)}{C_3^{11}} = \frac{46}{165}.
\]
21. Consider the following game, in which a referee picks a random integer between 1 and 100. One after the other, each of three players tries to guess the number the referee picked. Each player announces his or her guess before the next player guesses. Each guess has to be different from the previous guesses. The winner is the player who comes closest to the referee’s number without exceeding it. (It is possible for none of the players to win.)

Suppose that Player 1 guesses 24, and that Player 3 will guess a number that gives her/him the best chance of winning. What number should Player 2 guess to maximize his/her chances of winning?

(a) 1   (b) 25   (c) 62   (d) 63   (e) 64

Answer: (d)

Solution: First of all, observe that it always gives the third player the best chances to win if [s]he guesses the lowest number in some range which the other players left open. Thus, if you choose 63, this narrows the third player’s choices down to 1, 25, and 64. Guessing 25 gives a range of 38 numbers (25 through 62 inclusive) which win for the third player if the referee picked one of them, while a choice of 64 would only give 37 favorable numbers, and a choice of 1 only gives 23 favorable numbers.

Note that 63 gives you 38 favorable numbers also (63 through 100 inclusive) so this way you and the third player both have a 38% chance of winning, while the first player only has a 24% chance.

The other four given answers do not give you such a good result. Answer (e), 64, still makes 25 the best choice for the third player, but cuts your chances down to 37% while raising the third player’s to 39%. Answer (a), 1, is even worse than (e) because it only gives 23 chances for you. Answer (c), 62, would give the third player 63 as the best guess, (38% versus 37% if 25 is guessed instead) drastically reducing your winning chances to 1%. You would get the same baleful result if you guessed 25 [answer (b)], because then the obvious best guess for the third player is 26.
22. The Sierpiński Triangle involves a sequence of geometric figures. The first figure in the sequence is an equilateral triangle. The second has an inverted (shaded) equilateral triangle inscribed inside an equilateral triangle as shown. Each subsequent figure in this sequence is obtained by inserting an inverted (shaded) triangle inside each non-inverted (white) triangle of the previous figure, as shown below. How many regions (both shaded and white together) are in the ninth figure in this sequence?

For example, the first three figures in the sequence have 1 region, 4 regions, and 13 regions respectively.

(a) 4021  (b) 4022  (c) 4023  (d) 9841  (e) 9842

Answer: (d).

Solution: The number of regions is given by the recurrence \( R(1) = 1 \) and \( R(n) = 3R(n - 1) + 1 \) for \( n \geq 2 \). The value \( R(9) \) can then be obtained by iterating the recurrence or solving \( R(n) = (3^n - 1)/2 \).

23. How many positive integers \( n \) have the property that when 1,000,063 is divided by \( n \), the remainder is 63?

(a) 29  (b) 37  (c) 39  (d) 49  (e) 79

Answer: (b)

Solution: Suppose \( n \) is a positive integer such that the remainder when 1,000,063 is divided by \( n \) is 63. Then, 1,000,063 = \( nq + 63 \) where \( n > 63 \) (the remainder is always less than the divisor). Thus 1,000,000 = \( 10^6 = nq \). All we have to do is count the divisors of \( 10^6 \) that are greater than 63. Now \( 10^6 = 2^6 \times 5^6 \) has 49 positive divisors. Exactly twelve are \( \leq 63 \), namely 1, 2, 4, 8, 16, 32, 5, 10, 20, 40, 25, 50. Thus, the answer is 37.
24. I have twenty 3¢ stamps and twenty 5¢ stamps. Using one or more of these stamps, how many
different amounts of postage can I make?

(a) 150  (b) 152  (c) 154  (d) 396  (e) 400

Answer: (b)

Solution: We need to find how many positive integers can be represented in the form $3n + 5m$
where $n$ and $m$ are integers between 0 and 20, but not with both of them 0. First, let us see
which integers divisible by 3 can be represented in this form. These are 3, 6, 9, ···, 60 if we
set $m = 0$, and if we let $m$ be $3, 6, 9, 12, 15$, or 18, we get that the multiples of 3 we can
represent are $3, 6, 9, ···, 150$. There are thus 50 multiples of 3 we can represent. Working
similarly, we get that $5, 8, 11, ···, 155$ are the integers which give remainder 2 when divided
by 3, and which we can represent, giving 51 numbers of this form. Finally, the numbers which
give remainder 1 when divided by 3, and which can be represented in the desired form, are
$10, 13, 16, ···, 160$. There are 51 of them. So, the answer is $50 + 51 + 51 = 152$.

25. All of the positive integers are written in a triangular pattern, beginning with the following
four lines and continuing in the same way:

1
2  3  4
5  6  7  8  9
10 11 12 13 14 15 16

Which number appears directly below 2012?

(a) 2100  (b) 2102  (c) 2104  (d) 2106  (e) 2108

Answer: (b)

Solution: Observe that the $n$th row has $2n - 1$ numbers, that each number on the $2n$th row is
equal to $2n - 2$ plus the one above it, and that the last number in the $n$th row is $n^2$. (The latter
fact may be proved by observing that $n^2 - (n - 1)^2 = 2n - 1$, the length of the $n$th row.)

$44^2 = 1936$ and $45^2 = 2025$, so that 2012 is on the 45th row. The number below it will be
larger by $(2 \cdot 46 - 2)$, so it will be 2102.
26. Two spies agreed to meet at a gas station between noon and 1pm, but they have both forgotten the arranged time. Each arrives at a random time between noon and 1pm and stays for 6 minutes unless the other is there before the 6 minutes are up. Assuming all random times are equally likely, what is the probability that they will meet within the hour (noon to 1pm)?

(a) 0.12  (b) 0.15  (c) 0.17  (d) 0.19  (e) 0.25

**Answer:** (d)

**Solution:** The probability that they do not meet is represented by the total of the areas of the two outer triangles in the figure below, which is 0.81. So the probability of a meeting is 1 - 0.81 = .19.

27. A farmer has 12 plots of land, arranged in a row. To ensure viability of the soil, the farmer never uses two adjacent plots at the same time. This season, the farmer wishes to plant one plot of each of the following: corn, wheat, soybeans, and rice. Each crop is assigned its own plot of land. How many ways can the farmer allocate plots of land for these crops?

(a) 1680  (b) 3024  (c) 5040  (d) 7920  (e) 11880

**Answer:** (b)

**Solution:** There is a natural one-to-one correspondence between all choices from a row of 9 plots, and all “legal” choices for the 12 plots in the problem: given a choice of four plots in 9, insert a fallow plot after each of the first three plots. The number of ways to pick 4 plots of land from 9 is \( \binom{9}{4} \). After the locations of plots have been selected, there are 4! ways to assign the crops to the plots. The total number of ways is therefore 4!(\( \binom{9}{4} \)), or 3024.
28. How many triples \((x, y, z)\) of rational numbers satisfy the following system of equations?

\[
\begin{align*}
  x + y + z &= 0 \\
x y z + z &= 0 \\
x y + y z + x z + y &= 0
\end{align*}
\]

(a) 1  (b) 2  (c) 3  (d) 4  (e) 5

**Answer:** (b)

**Solution:**

If \(z = 0\), we have \(\begin{cases} x + y = 0 \\ x y + y = 0 \end{cases}\) then we can get the solutions \(\begin{cases} x = 0 \\ y = 0 \end{cases}\) or \(\begin{cases} x = -1 \\ y = 1 \end{cases}\).

If \(z \neq 0\), from \(x y z + z = 0\) we have

\[x y = -1 \quad \text{or} \quad x = \frac{-1}{y}\]  \hspace{1cm} (1)

From \(x + y + z = 0\) we have

\[z = -x - y\]  \hspace{1cm} (2)

Substituting (2) into \(x y + y z + x z + y = 0\), we can get

\[x^2 + y^2 + x y - y = 0\]  \hspace{1cm} (3)

Using (1), we substitute for \(x\) in (3) and multiply the result by \(y^2\), obtaining \(y^4 - y^3 + y^2 - 1 = (y - 1)(y^3 - y - 1) = 0\). Since \(y^3 - y - 1 = 0\) does not have rational solutions, we get \(y = 1\).

From (1), we have \(x = -1\). Then from (2), we have \(z = 0\). But this conflicts with \(z \neq 0\).

In summary, the above system has two sets of rational solutions

\[
\begin{cases}
  x = 0 & \quad \text{or} \quad x = -1 \\
y = 0 & \quad y = 1 \\
z = 0 & \quad z = 0
\end{cases}
\]
29. A coin has a probability of $\frac{1}{3}$ for coming up heads and $\frac{2}{3}$ for coming up tails. On average, how many flips of this coin are needed to guarantee both heads and tails appear at least once?

(a) 2.25  (b) 2.5  (c) 3  (d) 3.5  (e) 5

Answer: (d)

Solution: Let $H$ be the average number of times one must flip this coin to get heads to come up. Let $T$ be corresponding figure for tails. On the first flip, we get heads with probability $\frac{1}{3}$, and tails with probability $\frac{2}{3}$.

The average number of flips for getting heads, given that the first flip resulted in tails, is $H + 1$.

In trials in which the first flip gives tails, the corresponding figure for tails is $T + 1$. So:

$$H = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (H + 1).$$

Solving for $H$ we get $H = 3$. Similarly,

$$T = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot (T + 1).$$

So, $T = \frac{3}{2}$.

If we flip the coin and we get heads (with probability $\frac{1}{3}$), we will need $\frac{3}{2}$ more flips on average to get tails, so in one third of the cases we will need $\frac{5}{2}$ flips.

If we get tails on the first flip (with probability $\frac{2}{3}$), we will need 3 more flips on average to get heads, so in two thirds of the cases we will need 4 flips. Thus, on average, we will need $\frac{1}{3} \cdot \frac{5}{2} + \frac{2}{3} \cdot 4 = \frac{5}{6} + \frac{8}{3} = \frac{21}{6} = \frac{7}{2}$ flips.
30. Suppose \( a, b, \) and \( c \) are three successive terms in a geometric progression, and are also the lengths of the three sides opposite the angles \( A, B, \) and \( C, \) respectively, of \( \triangle ABC. \)

Which of the following intervals is the set of possible values of \( \frac{\sin A \cot C + \cos A}{\sin B \cot C + \cos B} \)?

(a) \((0, +\infty)\) (b) \(\left(0, \frac{\sqrt{5} + 1}{2}\right)\) (c) \(\left(\frac{\sqrt{5} - 1}{2}, \frac{\sqrt{5} + 1}{2}\right)\) (d) \(\left(\frac{\sqrt{5} - 1}{2}, +\infty\right)\) (e) \(\left(\frac{\sqrt{5} + 1}{2}, +\infty\right)\)

Answer: (c)

Solution: Suppose the ratio between \( a, b, c \) is \( r \), then \( b = ar \), and \( c = ar^2 \).

\[
\frac{\sin A \cot C + \cos A}{\sin B \cot C + \cos B} = \frac{\sin A \cos C + \cos A \sin C}{\sin B \cos C + \cos B \sin C} = \frac{\sin(A + C)}{\sin(B + C)} = \frac{\sin(\pi - B)}{\sin(\pi - A)} = \frac{\sin B}{\sin A} = \frac{b}{a} = r,
\]

so we just need to compute the range of \( r \).

Since \( a, b, c \) is a geometric sequence, the maximum length can only be \( a \) or \( c \). Also because \( a, b, c \) are the lengths of three sides of a triangle, they should satisfy \( a + b > c \) and \( b + c > a \).

So \( \begin{align*}
a + ar & > ar^2 \\
ar + ar^2 & > a
\end{align*} \), which is \( \begin{align*}
r^2 - r - 1 & < 0 \\
r^2 + r - 1 & > 0
\end{align*} \). Solving for \( r \), we find

\[
\begin{align*}
\frac{1 - \sqrt{5}}{2} & < r < \frac{\sqrt{5} + 1}{2} \\
r & > \frac{\sqrt{5} - 1}{2} \quad \text{or} \quad r < -\frac{\sqrt{5} + 1}{2}.
\end{align*}
\]

Therefore, the range of \( r \) is \( \left(\frac{\sqrt{5} - 1}{2}, \frac{\sqrt{5} + 1}{2}\right) \).