The number of pairs \((m, n)\) such that \(2^m - 2^n = 63\) in which \(m\) and \(n\) are nonnegative integers is

(a) 0  (b) 1  (c) 2  (d) 3  (e) more than 3
Problem # 1

The number of pairs \((m, n)\) such that \(2^m - 2^n = 63\) in which \(m\) and \(n\) are nonnegative integers is

(a) 0  (b) 1  (c) 2  (d) 3  (e) more than 3

The left-hand side is an even number unless \(m\) or \(n\) is zero.
The number of pairs \((m, n)\) such that \(2^m - 2^n = 63\) in which \(m\) and \(n\) are nonnegative integers is

(a) 0  (b) 1  (c) 2  (d) 3  (e) more than 3

- The left-hand side is an even number unless \(m\) or \(n\) is zero.
- If \(m = 0\), we have \(2^n \geq 1\) and \(2^m - 2^n \leq 0\).
The number of pairs \((m, n)\) such that \(2^m - 2^n = 63\) in which \(m\) and \(n\) are nonnegative integers is

(a) 0  (b) 1  (c) 2  (d) 3  (e) more than 3

- The left-hand side is an even number unless \(m\) or \(n\) is zero.
- If \(m = 0\), we have \(2^n \geq 1\) and \(2^m - 2^n \leq 0\).
- If \(n = 0\), then \(2^m = 64\), i.e. \(m = 6\).
The number of pairs \((m, n)\) such that \(2^m - 2^n = 63\) in which \(m\) and \(n\) are nonnegative integers is

(a) 0  (b) 1  (c) 2  (d) 3  (e) more than 3

- The left-hand side is an even number unless \(m\) or \(n\) is zero.
- If \(m = 0\), we have \(2^n \geq 1\) and \(2^m - 2^n \leq 0\).
- If \(n = 0\), then \(2^m = 64\), i.e. \(m = 6\).
- Thus \((6, 0)\) is the only such pair.
Problem # 2

How many real solutions does the following equation have?

\[
\left( \frac{2x^2 - 5}{3} \right)^{x^2-2x} = 1
\]

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4
Problem # 2

How many real solutions does the following equation have?

$$\left( \frac{2x^2 - 5}{3} \right)^{x^2-2x} = 1$$

(a) 0       (b) 1       (c) 2       (d) 3       (e) 4

Either \( \frac{2x^2 - 5}{3} = 1 \iff x = \pm 2 \)
Problem # 2

How many real solutions does the following equation have?

\[
\left( \frac{2x^2 - 5}{3} \right)^{x^2 - 2x} = 1
\]

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4

- Either \( \frac{2x^2 - 5}{3} = 1 \) ⇔ \( x = ±2 \)
- Or \( \frac{2x^2 - 5}{3} = -1 \) ⇔ \( x = ±1 \)

But then \( x^2 - 2x \) is odd and \( \left( \frac{2x^2 - 5}{3} \right)^{x^2 - 2x} = -1 \)
Problem # 2

How many real solutions does the following equation have?

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\left( \frac{2x^2 - 5}{3} \right)^{x^2 - 2x} = 1
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(a) 0  (b) 1  (c) 2  (d) 3  (e) 4

- Either \( \frac{2x^2 - 5}{3} = 1 \) \( \iff \) \( x = \pm 2 \)
- Or \( \frac{2x^2 - 5}{3} = -1 \) \( \iff \) \( x = \pm 1 \)

But then \( x^2 - 2x \) is odd and \( \left( \frac{2x^2 - 5}{3} \right)^{x^2 - 2x} = -1 \)

- Or \( x^2 - 2x = 0 \) \( \iff \) \( x = 0 \) or \( x = 2 \)
How many real solutions does the following equation have?

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\left( \frac{2x^2 - 5}{3} \right)^{x^2 - 2x} = 1
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(a) 0       (b) 1       (c) 2       (d) 3       (e) 4

- Either \( \frac{2x^2 - 5}{3} = 1 \) \( \iff \) \( x = \pm 2 \)
- Or \( \frac{2x^2 - 5}{3} = -1 \) \( \iff \) \( x = \pm 1 \)
  
  But then \( x^2 - 2x \) is odd and
  
  \[
  \left( \frac{2x^2 - 5}{3} \right)^{x^2 - 2x} = -1
  \]

- Or \( x^2 - 2x = 0 \) \( \iff \) \( x = 0 \) or \( x = 2 \)
- 3 solutions: \( x = 0, \pm 2 \)
Problem # 3

What is the remainder when $x^{51} + 51$ is divided by $x + 1$?

(a) 0  (b) 1  (c) 49  (d) 50  (e) 51
Problem # 3

What is the remainder when $x^{51} + 51$ is divided by $x + 1$?

(a) 0  (b) 1  (c) 49  (d) 50  (e) 51

- Write $x^{51} + 51 = Q(x) \cdot (x + 1) + R$, where $Q(x)$ is the quotient and $R$ is the remainder.
Problem # 3

What is the remainder when \( x^{51} + 51 \) is divided by \( x + 1 \)?

(a) 0  (b) 1  (c) 49  (d) 50  (e) 51

- Write \( x^{51} + 51 = Q(x) \cdot (x + 1) + R \), where \( Q(x) \) is the quotient and \( R \) is the remainder.
- Plug in \( x = -1 \). This gives \( R = (-1)^{51} + 51 = 50 \).
Problem #4

At a party, each person shakes hands with 5 other people. There are a total of 60 handshakes. How many people are at the party?

(a) 6   (b) 12   (c) 15   (d) 24   (e) 30
At a party, each person shakes hands with 5 other people. There are a total of 60 handshakes. How many people are at the party?

(a) 6  (b) 12  (c) 15  (d) 24  (e) 30

Let $x$ be the total number of people.
At a party, each person shakes hands with 5 other people. There are a total of 60 handshakes. How many people are at the party?

(a) 6  (b) 12  (c) 15  (d) 24  (e) 30

- Let \( x \) be the total number of people.
- The total number of handshakes is \( \frac{5x}{2} \)
Problem #4

At a party, each person shakes hands with 5 other people. There are a total of 60 handshakes. How many people are at the party?

(a) 6  (b) 12  (c) 15  (d) 24  (e) 30

- Let $x$ be the total number of people.
- The total number of handshakes is $\frac{5x}{2}$ (we have to divide by 2 since every handshake is counted twice)
At a party, each person shakes hands with 5 other people. There are a total of 60 handshakes. How many people are at the party?

(a) 6  (b) 12  (c) 15  (d) 24  (e) 30

- Let $x$ be the total number of people.
- The total number of handshakes is $\frac{5x}{2}$ (we have to divide by 2 since every handshake is counted twice).
- Thus $\frac{5x}{2} = 60 \Rightarrow x = 24$. 
Problem # 5

What is the units digit of $6^{25} - 3^{24}$?  
(a) 3  (b) 5  (c) 7  (d) 8  (e) none of the previous choices
What is the units digit of $6^{25} - 3^{24}$?

(a) 3  (b) 5  (c) 7  (d) 8  (e) none of the previous choices

Every power of six ends in 6.
Problem # 5

What is the units digit of $6^{25} - 3^{24}$?

(a) 3  (b) 5  (c) 7  (d) 8  (e) none of the previous choices

- Every power of six ends in 6.
- $3^{24} = 81^6$. 

Problem # 5

What is the units digit of $6^{25} - 3^{24}$?

(a) 3  (b) 5  (c) 7  (d) 8  (e) none of the previous choices

- Every power of six ends in 6.
- $3^{24} = 81^6$.
- Since the last digit of 81 is 1, any power of it will also end in 1.

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High School Math Contest 2011 Solutions
What is the units digit of $6^{25} - 3^{24}$?

(a) 3  (b) 5  (c) 7  (d) 8  (e) none of the previous choices

- Every power of six ends in 6.
- $3^{24} = 81^6$.
- Since the last digit of 81 is 1, any power of it will also end in 1.
- The units digit of $6^{25} - 3^{24}$ is $6 - 1 = 5$. 
A square is inscribed inside a circle. If $x$ is the circumference of the circle and $y$ is the perimeter of the inscribed square, then the ratio $\frac{x}{y}$ belongs to the interval

(a) $[0, \frac{1}{2})$  \hspace{1em} (b) $[\frac{1}{2}, 1)$  \hspace{1em} (c) $[1, \frac{3}{2})$  \hspace{1em} (d) $[\frac{3}{2}, 2)$  \hspace{1em} (e) $[2, \infty)$
A square is inscribed inside a circle. If $x$ is the circumference of the circle and $y$ is the perimeter of the inscribed square, then the ratio $\frac{x}{y}$ belongs to the interval

(a) $[0, 1/2)$  (b) $[1/2, 1)$  (c) $[1, 3/2)$  (d) $[3/2, 2)$  (e) $[2, \infty)$

- Obviously, $x > y$. 

\[ x = 2\pi r \quad \text{and} \quad y = 4\sqrt{2}r. \]

\[ \frac{x}{y} = \frac{\pi}{\sqrt{2}} < \frac{3}{2}.8 \approx 1.14 < \frac{3}{2}.8. \]
A square is inscribed inside a circle. If $x$ is the circumference of the circle and $y$ is the perimeter of the inscribed square, then the ratio $\frac{x}{y}$ belongs to the interval

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A square is inscribed inside a circle. If $x$ is the circumference of the circle and $y$ is the perimeter of the inscribed square, then the ratio $\frac{x}{y}$ belongs to the interval

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- Obviously, $x > y$.
- If $r$ is the radius of the circle, then $x = 2\pi r$ and $y = 4\sqrt{2}r$.
- $\frac{x}{y} = \frac{\pi}{2\sqrt{2}} < \frac{3.2}{2.8} \approx 1.14 < \frac{3}{2}$. 

How many positive integers $x$ satisfy the inequality

$$\left( x - \frac{1}{2} \right)^1 \left( x - \frac{3}{2} \right)^3 \cdots \left( x - \frac{4021}{2} \right)^{4021} < 0?$$

(a) 503  (b) 999  (c) 1005  (d) 1006  (e) 1995
How many positive integers $x$ satisfy the inequality

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\left(x - \frac{1}{2}\right)^1 \left(x - \frac{3}{2}\right)^3 \cdots \left(x - \frac{4021}{2}\right)^{4021} < 0?
\]

(a) 503   (b) 999   (c) 1005   (d) 1006   (e) 1995

- The product has 2011 factors.
Problem # 7

How many positive integers \( x \) satisfy the inequality

\[
\left(x - \frac{1}{2}\right)^1 \left(x - \frac{3}{2}\right)^3 \cdots \left(x - \frac{4021}{2}\right)^{4021} < 0?
\]

(a) 503 (b) 999 (c) 1005 (d) 1006 (e) 1995

- The product has 2011 factors.
- We need an odd number of negative factors \( \Rightarrow \) an even number of positive factors.
How many positive integers $x$ satisfy the inequality
\[
\left( x - \frac{1}{2} \right)^1 \left( x - \frac{3}{2} \right)^3 \cdots \left( x - \frac{4021}{2} \right)^{4021} < 0?
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(a) 503  (b) 999  (c) 1005  (d) 1006  (e) 1995

- The product has 2011 factors.
- We need an odd number of negative factors $\Rightarrow$ an even number of positive factors.
- The number of positive factors is simply equal to $x$. 
How many positive integers $x$ satisfy the inequality

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\left( x - \frac{1}{2} \right)^1 \left( x - \frac{3}{2} \right)^3 \cdots \left( x - \frac{4021}{2} \right)^{4021} < 0?
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(a) 503    (b) 999    (c) 1005    (d) 1006    (e) 1995

- The product has 2011 factors.
- We need an odd number of negative factors $\Rightarrow$ an even number of positive factors.
- The number of positive factors is simply equal to $x$.
- There are 1005 even positive integers less than 2011.
How many positive integers $x$ satisfy the inequality

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\left( x - \frac{1}{2} \right)^1 \left( x - \frac{3}{2} \right)^3 \cdots \left( x - \frac{4021}{2} \right)^{4021} < 0?
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(a) 503  (b) 999  (c) 1005  (d) 1006  (e) 1995

- The product has 2011 factors.
- We need an odd number of negative factors $\Rightarrow$ an even number of positive factors.
- The number of positive factors is simply equal to $x$.
- There are 1005 even positive integers less than 2011.
  (For integers $x \geq 2011$ the product is obviously positive.)
Problem # 8

Let ABCD be a trapezoid with the sides $\overline{AD}$ and $\overline{BC}$ parallel to each other and perpendicular to $\overline{AB}$. Moreover, we are told that $BC = 2AD$. The points $P$ and $Q$ on the side $\overline{AB}$ divide it into three equal segments, and $P$ is between $A$ and $Q$. Similarly the points $R$ and $S$ on the side $\overline{CD}$ divide it into three equal segments, and $R$ is between $D$ and $S$. Let $a_1$ be the area of the triangle $\triangle AQR$, and let $a_2$ be the area of the triangle $\triangle PBS$. Find $a_1/a_2$.

(a) 1/2  (b) 2/3  (c) 4/5  (d) 5/4  (e) 3/2
Let ABCD be a trapezoid with the sides $\overline{AD}$ and $\overline{BC}$ parallel to each other and perpendicular to $\overline{AB}$. Moreover, we are told that $BC = 2AD$. The points $P$ and $Q$ on the side $\overline{AB}$ divide it into three equal segments, and $P$ is between $A$ and $Q$. Similarly the points $R$ and $S$ on the side $\overline{CD}$ divide it into three equal segments, and $R$ is between $D$ and $S$. Let $a_1$ be the area of the triangle $\triangle AQR$, and let $a_2$ be the area of the triangle $\triangle PBS$. Find $a_1/a_2$.

(a) $1/2$  (b) $2/3$  (c) $4/5$  (d) $5/4$  (e) $3/2$
Problem # 9

If \( x = \log_9 2 \) and \( y = \log_5 4 \), find \( \log_6 15 \) in terms of \( x \) and \( y \).

(a) \( \frac{2x+y}{xy} \)  
(b) \( \frac{2x+y}{y(1+4x)} \)  
(c) \( \frac{4x+y}{y(1+2x)} \)  
(d) \( \frac{y+4x}{2} \)  
(e) \( \frac{x(4x+y)}{y+1} \)
Problem # 9

If $x = \log_9 2$ and $y = \log_5 4$, find $\log_6 15$ in terms of $x$ and $y$.

(a) $\frac{2x+y}{xy}$  (b) $\frac{2x+y}{y(1+4x)}$  (c) $\frac{4x+y}{y(1+2x)}$  (d) $\frac{y+4x}{2}$  (e) $\frac{x(4x+y)}{y+1}$

- $x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2\log_2 3} = \frac{1}{2} \log_3 2$, so $\log_3 2 = 2x$. 

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Problem # 9

If $x = \log_9 2$ and $y = \log_5 4$, find $\log_6 15$ in terms of $x$ and $y$.

(a) $\frac{2x+y}{xy}$  
(b) $\frac{2x+y}{y(1+4x)}$  
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(d) $\frac{y+4x}{2}$  
(e) $\frac{x(4x+y)}{y+1}$

- $x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2\log_2 3} = \frac{1}{2} \log_3 2$, so $\log_3 2 = 2x$.
- $\log_5 2 = \frac{1}{2} \log_5 4 = \frac{1}{2} y$. 

University of South Carolina  High School Math Contest 2011 Solutions
Problem # 9

If $x = \log_9 2$ and $y = \log_5 4$, find $\log_6 15$ in terms of $x$ and $y$.

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(d) $\frac{y+4x}{2}$  
(e) $\frac{x(4x+y)}{y+1}$

- $x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2\log_2 3} = \frac{1}{2} \log_3 2$, so $\log_3 2 = 2x$.
- $\log_5 2 = \frac{1}{2} \log_5 4 = \frac{1}{2} y$.
- $\log_5 3 = \frac{\log_2 3}{\log_2 5} = \frac{\log_5 2}{\log_3 2} = \frac{y}{4x}$. 
Problem # 9

If \( x = \log_9 2 \) and \( y = \log_5 4 \), find \( \log_6 15 \) in terms of \( x \) and \( y \).

(a) \( \frac{2x+y}{xy} \)  
(b) \( \frac{2x+y}{y(1+4x)} \)  
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- \( x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2\log_2 3} = \frac{1}{2} \log_3 2 \), so \( \log_3 2 = 2x \).
- \( \log_5 2 = \frac{1}{2} \log_5 4 = \frac{1}{2} y \).
- \( \log_5 3 = \frac{\log_2 3}{\log_2 5} = \frac{\log_5 2}{\log_3 2} = \frac{y}{4x} \).

Hence

\[
\log_6 15 = \log_6 3 + \log_6 5 = \frac{1}{\log_3 6} + \frac{1}{\log_5 6}
\]
Problem # 9

If \( x = \log_9 2 \) and \( y = \log_5 4 \), find \( \log_6 15 \) in terms of \( x \) and \( y \).

(a) \( \frac{2x+y}{xy} \)  (b) \( \frac{2x+y}{y(1+4x)} \)  (c) \( \frac{4x+y}{y(1+2x)} \)  (d) \( \frac{y+4x}{2} \)  (e) \( \frac{x(4x+y)}{y+1} \)

- \( x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2 \log_2 3} = \frac{1}{2} \log_3 2 \), so \( \log_3 2 = 2x \).
- \( \log_5 2 = \frac{1}{2} \log_5 4 = \frac{1}{2} y \).
- \( \log_5 3 = \frac{\log_2 3}{\log_2 5} = \frac{\log_5 2}{\log_3 2} = \frac{y}{4x} \).

Hence

\[
\log_6 15 = \log_6 3 + \log_6 5 = \frac{1}{\log_3 6} + \frac{1}{\log_5 6}
\]

\[
= \frac{1}{\log_3 2 + \log_3 3} + \frac{1}{\log_5 2 + \log_5 3}
\]
If \( x = \log_9 2 \) and \( y = \log_5 4 \), find \( \log_6 15 \) in terms of \( x \) and \( y \).

(a) \( \frac{2x+y}{xy} \)  
(b) \( \frac{2x+y}{y(1+4x)} \)  
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- \( x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2 \log_2 3} = \frac{1}{2} \log_3 2 \), so \( \log_3 2 = 2x \).
- \( \log_5 2 = \frac{1}{2} \log_5 4 = \frac{1}{2}y \).
- \( \log_5 3 = \frac{\log_2 3}{\log_2 5} = \frac{\log_5 2}{\log_5 2} = \frac{y}{4x} \).

Hence

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\log_6 15 = \log_6 3 + \log_6 5 = \frac{1}{\log_3 6} + \frac{1}{\log_5 6}
\]

\[
= \frac{1}{\log_3 2 + \log_3 3} + \frac{1}{\log_5 2 + \log_5 3}
\]

\[
= \frac{1}{2x+1} + \frac{1}{2y + \frac{y}{4x}} = \frac{1}{2x+1} + \frac{4x}{y(2x + 1)}
\]
If \( x = \log_9 2 \) and \( y = \log_5 4 \), find \( \log_6 15 \) in terms of \( x \) and \( y \).

(a) \( \frac{2x+y}{xy} \)  
(b) \( \frac{2x+y}{y(1+4x)} \)  
(c) \( \frac{4x+y}{y(1+2x)} \)  
(d) \( \frac{y+4x}{2} \)  
(e) \( \frac{x(4x+y)}{y+1} \)

\[ x = \log_9 2 = \frac{1}{\log_2 9} = \frac{1}{2\log_2 3} = \frac{1}{2} \log_3 2, \text{ so } \log_3 2 = 2x. \]

\[ \log_5 2 = \frac{1}{2} \log_5 4 = \frac{1}{2} y. \]

\[ \log_5 3 = \frac{\log_2 3}{\log_2 5} = \frac{\log_5 2}{\log_3 2} = \frac{y}{4x}. \]

Hence

\[ \log_6 15 = \log_6 3 + \log_6 5 = \frac{1}{\log_3 6} + \frac{1}{\log_5 6} \]

\[ = \frac{1}{\log_3 2 + \log_3 3} + \frac{1}{\log_5 2 + \log_5 3} \]

\[ = \frac{1}{2x + 1} + \frac{1}{\frac{1}{2} y + \frac{y}{4x}} = \frac{1}{2x + 1} + \frac{4x}{y(2x + 1)} \]

\[ = \frac{4x + y}{y(2x + 1)}. \]
The irrational number $\sqrt{29} - 12\sqrt{5}$ can be expressed in the form $a + b\sqrt{n}$, where $a$ and $b$ are integers and $n$ is a positive integer. One possible set of values for the triple $(a, b, n)$ is:

(a) $(2, -4, 5)$  
(b) $(-2, 5, 3)$  
(c) $(-3, 7, 2)$  
(d) $(-3, 2, 5)$  
(e) $(3, 4, 5)$
The irrational number $\sqrt{29} - 12\sqrt{5}$ can be expressed in the form $a + b\sqrt{n}$, where $a$ and $b$ are integers and $n$ is a positive integer. One possible set of values for the triple $(a, b, n)$ is:

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(c) $(-3, 7, 2)$  
(d) $(-3, 2, 5)$  
(e) $(3, 4, 5)$

Write $\sqrt{29} - 12\sqrt{5} = a + b\sqrt{n}$ and square both sides.

$29 - 12\sqrt{5} = (a^2 + n)(b^2 + n)$

Take $n = 5$. Find $a$ and $b$:

$a^2 + 5b^2 = 29$ and $2ab = -12$.

$a = -3$ and $b = 2$.

(In fact, $a = 3$ and $b = -2$ also satisfy the equations, but in this case the number $a + b\sqrt{n}$ is negative.)
The irrational number $\sqrt{29} - 12\sqrt{5}$ can be expressed in the form $a + b\sqrt{n}$, where $a$ and $b$ are integers and $n$ is a positive integer. One possible set of values for the triple $(a, b, n)$ is:

(a) (2, $-4$, 5)  
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(c) ($-3$, 7, 2)  
(d) ($-3$, 2, 5)  
(e) (3, 4, 5)

- Write $\sqrt{29} - 12\sqrt{5} = a + b\sqrt{n}$ and square both sides.
- $29 - 12\sqrt{5} = (a^2 + b^2n) + 2ab\sqrt{n}$
The irrational number $\sqrt{29} - 12\sqrt{5}$ can be expressed in the form $a + b\sqrt{n}$, where $a$ and $b$ are integers and $n$ is a positive integer. One possible set of values for the triple $(a, b, n)$ is:

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- Write $\sqrt{29} - 12\sqrt{5} = a + b\sqrt{n}$ and square both sides.
- $29 - 12\sqrt{5} = (a^2 + b^2n) + 2ab\sqrt{n}$
- Take $n = 5$. Find $a$ and $b$: $a^2 + 5b^2 = 29$ and $2ab = -12$. 
The irrational number $\sqrt{29} - 12\sqrt{5}$ can be expressed in the form $a + b\sqrt{n}$, where $a$ and $b$ are integers and $n$ is a positive integer. One possible set of values for the triple $(a, b, n)$ is:

(a) $(2, -4, 5)$  
(b) $(-2, 5, 3)$  
(c) $(-3, 7, 2)$  
(d) $(-3, 2, 5)$  
(e) $(3, 4, 5)$

- Write $\sqrt{29} - 12\sqrt{5} = a + b\sqrt{n}$ and square both sides.
- $29 - 12\sqrt{5} = (a^2 + b^2n) + 2ab\sqrt{n}$
- Take $n = 5$. Find $a$ and $b$: $a^2 + 5b^2 = 29$ and $2ab = -12$.
- $a = -3$ and $b = 2$. 
The irrational number $\sqrt{29} - 12\sqrt{5}$ can be expressed in the form $a + b\sqrt{n}$, where $a$ and $b$ are integers and $n$ is a positive integer. One possible set of values for the triple $(a, b, n)$ is:

(a) $(2, -4, 5)$ \hspace{1cm} (b) $(-2, 5, 3)$ \hspace{1cm} (c) $(-3, 7, 2)$

(d) $(-3, 2, 5)$ \hspace{1cm} (e) $(3, 4, 5)$

- Write $\sqrt{29} - 12\sqrt{5} = a + b\sqrt{n}$ and square both sides.
- $29 - 12\sqrt{5} = (a^2 + b^2n) + 2ab\sqrt{n}$
- Take $n = 5$. Find $a$ and $b$: $a^2 + 5b^2 = 29$ and $2ab = -12$.
- $a = -3$ and $b = 2$. (In fact, $a = 3$ and $b = -2$ also satisfy the equations, but in this case the number $a + b\sqrt{n}$ is negative.)
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89     (b) 144     (c) 243     (d) 512     (e) 1024
Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89  (b) 144  (c) 243  (d) 512  (e) 1024

Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89  (b) 144  (c) 243  (d) 512  (e) 1024

- Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
- Student with the ticket to seat number $n$ has two options:
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89  (b) 144  (c) 243  (d) 512  (e) 1024

- Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
- Student with the ticket to seat number $n$ has two options:
  - A) sit in his/her own chair.
  - B) take the $(n-1)$st seat. $(n-1)$st student is forced to take the $n$th seat. Other $n-2$ students can sit in $F(n-2)$ ways.

$$F(n) = F(n-1) + F(n-2)$$

Fibonacci numbers!!!

$F(1) = 1$, $F(2) = 2$  \[\Rightarrow F(10) = 89\]
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89    (b) 144    (c) 243    (d) 512    (e) 1024

- Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
- Student with the ticket to seat number $n$ has two options: A) sit in his/her own chair.
- Remaining $n - 1$ students can sit in $F(n - 1)$ ways.
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89    (b) 144    (c) 243    (d) 512    (e) 1024

- Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
- Student with the ticket to seat number $n$ has two options:
  - A) sit in his/her own chair.
    - Remaining $n - 1$ students can sit in $F(n - 1)$ ways.
  - B) take the $(n - 1)^{st}$ seat.
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89  (b) 144  (c) 243  (d) 512  (e) 1024

Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.

Student with the ticket to seat number $n$ has two options:
A) sit in his/her own chair.
Remaining $n - 1$ students can sit in $F(n - 1)$ ways.
B) take the $(n - 1)^{st}$ seat.
$(n - 1)^{st}$ student is forced to take the $n^{th}$ seat.
Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89  (b) 144  (c) 243  (d) 512  (e) 1024

- Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
- Student with the ticket to seat number $n$ has two options:
  - A) sit in his/her own chair.
  - Remaining $n - 1$ students can sit in $F(n - 1)$ ways.
  - B) take the $(n - 1)^{st}$ seat.
  - $(n - 1)^{st}$ student is forced to take the $n^{th}$ seat.
  - Other $n - 2$ students can sit in $F(n - 2)$ ways.
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89    (b) 144    (c) 243    (d) 512    (e) 1024

Let \( F(n) \) be the number of ways to seat \( n \) students in a row of \( n \) chairs as described in the problem.

Student with the ticket to seat number \( n \) has two options:

A) sit in his/her own chair.

Remaining \( n - 1 \) students can sit in \( F(n - 1) \) ways.

B) take the \((n - 1)^{st}\) seat.

\((n - 1)^{st}\) student is forced to take the \( n^{th} \) seat.

Other \( n - 2 \) students can sit in \( F(n - 2) \) ways.

\[ F(n) = F(n - 1) + F(n - 2) \] Fibonacci numbers!!!
Problem # 11

Each of 10 students has a ticket to one of ten chairs in a row at a theater. How many ways are there to seat the students so that each student sits either in the chair specified on his/her ticket or in one right next to it? Each chair is to be occupied by exactly one student.

(a) 89  (b) 144  (c) 243  (d) 512  (e) 1024

- Let $F(n)$ be the number of ways to seat $n$ students in a row of $n$ chairs as described in the problem.
- Student with the ticket to seat number $n$ has two options:
  - A) sit in his/her own chair.
    - Remaining $n - 1$ students can sit in $F(n - 1)$ ways.
  - B) take the $(n - 1)^{st}$ seat.
    - $(n - 1)^{st}$ student is forced to take the $n^{th}$ seat.
    - Other $n - 2$ students can sit in $F(n - 2)$ ways.
- $F(n) = F(n - 1) + F(n - 2)$ Fibonacci numbers!!!
- $F(1) = 1$, $F(2) = 2$ $\Rightarrow$ $F(10) = 89$
Problem # 12

Evaluate the sum
\[ \ln(\tan 1^\circ) + \ln(\tan 2^\circ) + \cdots + \ln(\tan 88^\circ) + \ln(\tan 89^\circ). \]

(a) 0  (b) 1  (c) 2  (d) \( \ln \left( \frac{\pi}{2} \right) \)  (e) \( \ln \left( \frac{\pi}{4} \right) \)
Evaluate the sum
\[ \ln (\tan 1^\circ) + \ln (\tan 2^\circ) + \cdots + \ln (\tan 88^\circ) + \ln (\tan 89^\circ). \]
(a) 0 (b) 1 (c) 2 (d) \( \ln \left( \frac{\pi}{2} \right) \) (e) \( \ln \left( \frac{\pi}{4} \right) \)

For \( 0 < x < 90^\circ \), we have
\[ \ln (\tan(90^\circ - x)) = \ln \left( \frac{1}{\tan x} \right) = -\ln (\tan x). \]
Evaluate the sum
\[ \ln (\tan 1^\circ) + \ln (\tan 2^\circ) + \cdots + \ln (\tan 88^\circ) + \ln (\tan 89^\circ). \]

(a) 0  (b) 1  (c) 2  (d) \( \ln \left(\frac{\pi}{2}\right) \)  (e) \( \ln \left(\frac{\pi}{4}\right) \)

- For \( 0 < x < 90^\circ \), we have
  \[ \ln (\tan(90^\circ - x)) = \ln \left(\frac{1}{\tan x}\right) = -\ln (\tan x). \]

- \( \ln (\tan 1^\circ) + \ln (\tan 89^\circ) = \ln (\tan 2^\circ) + \ln (\tan 88^\circ) = \ln (\tan 45^\circ) = 0 \), so the sum is zero.
In the figure below, $ABCD$ is a rectangle. The points $A$, $F$, and $E$ lie on a straight line. The segments $DF$, $BE$, and $CA$ are all perpendicular to $FE$. Denote the length of $DF$ by $a$ and the length of $BE$ by $b$. Find the length of $FE$ in terms of $a$ and $b$.

(a) $a + b$  
(b) $2\sqrt{ab}$  
(c) $\sqrt{2a^2 + 2b^2}$  
(d) $\frac{2a + 4b}{3}$  
(e) none of the previous choices
In the figure below, $ABCD$ is a rectangle. The points $A$, $F$, and $E$ lie on a straight line. The segments $DF$, $BE$, and $CA$ are all perpendicular to $FE$. Denote the length of $DF$ by $a$ and the length of $BE$ by $b$. Find the length of $FE$ in terms of $a$ and $b$.

(a) $a + b$  
(b) $2\sqrt{ab}$  
(c) $\sqrt{2a^2 + 2b^2}$  
(d) $\frac{2a + 4b}{3}$

(e) none of the previous choices
\[
\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right) = \frac{x}{2 \cdot 2011}
\]

What is the value of \(x\)?

(a) 1  (b) 2010  (c) 2011  (d) 2012  (e) none of the previous choices
Problem # 14

\[
\left( 1 - \frac{1}{2^2} \right) \left( 1 - \frac{1}{3^2} \right) \cdots \left( 1 - \frac{1}{2011^2} \right) = \frac{x}{2 \cdot 2011}
\]

What is the value of \( x \)?

(a) 1 (b) 2010 (c) 2011 (d) 2012 (e) none of the previous choices

\[
1 - \frac{1}{k^2} = \frac{k^2 - 1}{k^2} = \frac{(k - 1)(k + 1)}{k^2}
\]
Problem # 14

\[
\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right) = \frac{x}{2 \cdot 2011}
\]

What is the value of \(x\)?

(a) 1  
(b) 2010  
(c) 2011  
(d) 2012  
(e) none of the previous choices

\[
1 - \frac{1}{k^2} = \frac{k^2 - 1}{k^2} = \frac{(k - 1)(k + 1)}{k^2}
\]

Thus

\[
\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right) = \frac{x}{2 \cdot 2011}
\]
Problem # 14

\[
\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right) = \frac{x}{2 \cdot 2011}
\]

What is the value of \(x\)?

(a) 1 \hspace{1cm} (b) 2010 \hspace{1cm} (c) 2011 \hspace{1cm} (d) 2012

(e) none of the previous choices

\[
1 - \frac{1}{k^2} = \frac{k^2 - 1}{k^2} = \frac{(k - 1)(k + 1)}{k^2}
\]

Thus

\[
\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{2011^2}\right) = \frac{1 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 5 \cdots \cdot 2009 \cdot 2011 \cdot 2010 \cdot 2012}{2^2 \cdot 3^2 \cdot 4^2 \cdots \cdot 2010^2 \cdot 2011^2} = \frac{2012}{2 \cdot 2011}.
\]
Problem # 15

Allan and Bill are walking in the same direction beside a railroad track, and Allan is far behind Bill. Both walk at constant speeds, and Allan walks faster than Bill. A long train traveling at a constant speed in the same direction will take 10 seconds to pass Allan (from the front to the end) and will take 9 seconds to pass Bill. If it will take twenty minutes for the front of the train to travel from Allan to Bill, how many minutes will it take for Allan to catch up to Bill?

(a) 200      (b) 220      (c) 240      (d) 260      (e) 280
Allan and Bill are walking in the same direction beside a railroad track, and Allan is far behind Bill. Both walk at constant speeds, and Allan walks faster than Bill. A long train traveling at a constant speed in the same direction will take 10 seconds to pass Allan (from the front to the end) and will take 9 seconds to pass Bill. If it will take twenty minutes for the front of the train to travel from Allan to Bill, how many minutes will it take for Allan to catch up to Bill?

(a) 200  (b) 220  (c) 240  (d) 260  (e) 280

\[
\frac{v_T - v_A}{v_T - v_B} = \frac{9}{10}.
\]
Allan and Bill are walking in the same direction beside a railroad track, and Allan is far behind Bill. Both walk at constant speeds, and Allan walks faster than Bill. A long train traveling at a constant speed in the same direction will take 10 seconds to pass Allan (from the front to the end) and will take 9 seconds to pass Bill. If it will take twenty minutes for the front of the train to travel from Allan to Bill, how many minutes will it take for Allan to catch up to Bill?

(a) 200  (b) 220  (c) 240  (d) 260  (e) 280

\[
\frac{v_T - v_A}{v_T - v_B} = \frac{9}{10}.
\]

\[
\frac{v_A - v_B}{v_T - v_B} = \frac{v_T - v_B}{v_T - v_B} - \frac{v_T - v_A}{v_T - v_B} = 1 - \frac{9}{10} = \frac{1}{10}.
\]
Allan and Bill are walking in the same direction beside a railroad track, and Allan is far behind Bill. Both walk at constant speeds, and Allan walks faster than Bill. A long train traveling at a constant speed in the same direction will take 10 seconds to pass Allan (from the front to the end) and will take 9 seconds to pass Bill. If it will take twenty minutes for the front of the train to travel from Allan to Bill, how many minutes will it take for Allan to catch up to Bill?

(a) 200  (b) 220  (c) 240  (d) 260  (e) 280

\[
\frac{v_T - v_A}{v_T - v_B} = \frac{9}{10}.
\]
\[
\frac{v_A - v_B}{v_T - v_B} = \frac{v_T - v_B}{v_T - v_B} - \frac{v_T - v_A}{v_T - v_B} = 1 - \frac{9}{10} = \frac{1}{10}.
\]

Allan’s speed with respect to Bill is ten times smaller than the train’s, so, in order to reach Bill, it would take Allan ten times the time it would take the train, i.e. 200 minutes.
Problem # 16

What is the remainder when $10^{20}$ is divided by 1001?

(a) 1  
(b) 100  
(c) 1000  
(d) 999  
(e) 10
Problem # 16

What is the remainder when $10^{20}$ is divided by 1001?

(a) 1    (b) 100    (c) 1000    (d) 999    (e) 10

$10^{20} = 10^{3 \cdot 6 + 2}$
Problem # 16

What is the remainder when \(10^{20}\) is divided by 1001?

(a) 1  (b) 100  (c) 1000  (d) 999  (e) 10

\[10^{20} = 10^{3\cdot6+2}\]

\[10^3 = 1001 - 1 \quad \Rightarrow \quad 10^{3\cdot6} = (1001 - 1)^6 = 1001 \cdot k + 1\]
What is the remainder when \(10^{20}\) is divided by 1001?

(a) 1  
(b) 100  
(c) 1000  
(d) 999  
(e) 10

\[10^{20} = 10^3 \cdot 6 + 2\]

\[10^3 = 1001 - 1 \implies 10^3 \cdot 6 = (1001 - 1)^6 = 1001 \cdot k + 1\]

\[10^{20} = (1001 \cdot k + 1) \cdot 100 = 1001 \cdot 100k + 100\]
What is the remainder when \(10^{20}\) is divided by 1001?

(a) 1  (b) 100  (c) 1000  (d) 999  (e) 10

\[
10^{20} = 10^3 \cdot 6 + 2
\]

\[
10^3 = 1001 - 1 \quad \Rightarrow \quad 10^3 \cdot 6 = (1001 - 1)^6 = 1001 \cdot k + 1
\]

\[
10^{20} = (1001 \cdot k + 1) \cdot 100 = 1001 \cdot 100k + 100
\]

It has remainder 100 when divided by 1001.
If \( f(x) = \left( \frac{1}{2} - x \right)^{x-\frac{1}{2}} \), which of the following is the largest?

(a) \( f(-1/2) \)  
(b) \( f(0) \)  
(c) \( f(-1) \)  
(d) \( f(-3/2) \)  
(e) \( f(1/3) \)
Problem # 17

If \( f(x) = \left( \frac{1}{2} - x \right)^{x-\frac{1}{2}} \), which of the following is the largest?

(a) \( f(-1/2) \)  
(b) \( f(0) \)  
(c) \( f(-1) \)  
(d) \( f(-3/2) \)  
(e) \( f(1/3) \)

\[ f(-1/2) = 1 \]
If \( f(x) = \left( \frac{1}{2} - x \right)^{x-\frac{1}{2}} \), which of the following is the largest?

(a) \( f(-1/2) \)  
(b) \( f(0) \)  
(c) \( f(-1) \)  
(d) \( f(-3/2) \)  
(e) \( f(1/3) \)

- \( f(-1/2) = 1 \)
- \( f(-1) = (3/2)^{-\frac{3}{2}} < 1 \) and \( f(-3/2) = 2^{-2} = 1/4 < 1 \).
If \( f(x) = \left( \frac{1}{2} - x \right)^{x^{-\frac{1}{2}}} \), which of the following is the largest?

(a) \( f(-\frac{1}{2}) \)  
(b) \( f(0) \)  
(c) \( f(-1) \)  
(d) \( f(-\frac{3}{2}) \)  
(e) \( f(\frac{1}{3}) \)

- \( f(-\frac{1}{2}) = 1 \)
- \( f(-1) = (\frac{3}{2})^{-\frac{3}{2}} < 1 \) and \( f(-\frac{3}{2}) = 2^{-2} = 1/4 < 1 \).
- \( f(0) = (\frac{1}{2})^{-\frac{1}{2}} = \sqrt{2} \) and \( f(\frac{1}{3}) = (\frac{1}{6})^{-\frac{1}{6}} = \sqrt[6]{6} \)
  (both greater than 1)
Problem # 17

If \( f(x) = \left(\frac{1}{2} - x\right)^{x - \frac{1}{2}} \), which of the following is the largest?

(a) \( f(-\frac{1}{2}) \)  (b) \( f(0) \)  (c) \( f(-1) \)  (d) \( f(-\frac{3}{2}) \)  (e) \( f(\frac{1}{3}) \)

- \( f(-\frac{1}{2}) = 1 \)
- \( f(-1) = (\frac{3}{2})^{-\frac{3}{2}} < 1 \) and \( f(-\frac{3}{2}) = 2^{-2} = 1/4 < 1 \).
- \( f(0) = (\frac{1}{2})^{-\frac{1}{2}} = \sqrt{2} \) and \( f(\frac{1}{3}) = (\frac{1}{6})^{-\frac{1}{6}} = \sqrt[6]{6} \) (both greater than 1)
- \( 6 < 8 \implies 6^{\frac{1}{6}} < 8^{\frac{1}{6}} = 2^{\frac{1}{2}} \).
When a positive integer $N$ having two digits is multiplied by 1111, the answer can have five digits or six digits. Find the sum of the digits of $N$ if $N$ is the smallest two digit number with the property that $N \cdot 1111$ is a six digit number.

(a) 5     (b) 9     (c) 10     (d) 12     (e) 15
When a positive integer $N$ having two digits is multiplied by 1111, the answer can have five digits or six digits. Find the sum of the digits of $N$ if $N$ is the smallest two digit number with the property that $N \cdot 1111$ is a six digit number.

(a) 5 (b) 9 (c) 10 (d) 12 (e) 15

- $90 \cdot 1111 = 99990$ has 5 digits.
- $91 \cdot 1111 = 101101$ is a six-digit number.

The answer is 9 + 1 = 10.
Problem # 18

When a positive integer $N$ having two digits is multiplied by 1111, the answer can have five digits or six digits. Find the sum of the digits of $N$ if $N$ is the smallest two digit number with the property that $N \cdot 1111$ is a six digit number.

(a) 5  (b) 9  (c) 10  (d) 12  (e) 15

- $90 \cdot 1111 = 99990$ has 5 digits.
- $91 \cdot 1111 = 101101$ is a six-digit number.
When a positive integer $N$ having two digits is multiplied by 1111, the answer can have five digits or six digits. Find the sum of the digits of $N$ if $N$ is the smallest two digit number with the property that $N \cdot 1111$ is a six digit number.

(a) 5   (b) 9   (c) 10   (d) 12   (e) 15

- $90 \cdot 1111 = 99990$ has 5 digits.
- $91 \cdot 1111 = 101101$ is a six-digit number.
- The answer is $9 + 1 = 10$. 
Problem # 19

Let \( \triangle ABC \) be a right triangle with hypotenuse \( \overline{AB} \) and with the measure of \( \angle BAC \) equal to 32°. A square with side \( \overline{AB} \) is placed so that the interior of the square does not overlap with the interior of \( \triangle ABC \). Let \( P \) be the center of the square. What is the measure of \( \angle PCB \)?

(a) 30°  (b) 32°  (c) 45°  (d) 58°  (e) 60°
Problem # 19
Let $\triangle ABC$ be a right triangle with hypotenuse $AB$ and with the measure of $\angle BAC$ equal to $32^\circ$. A square with side $AB$ is placed so that the interior of the square does not overlap with the interior of $\triangle ABC$. Let $P$ be the center of the square. What is the measure of $\angle PCB$?

(a) $30^\circ$  (b) $32^\circ$  (c) $45^\circ$  (d) $58^\circ$  (e) $60^\circ$
How many positive integers less than one billion (i.e. $10^9$) are divisible by 9 and have all digits equal?

(a) 10  (b) 14  (c) 18  (d) 21  (e) 25
How many positive integers less than one billion (i.e. $10^9$) are divisible by 9 and have all digits equal?

(a) 10  (b) 14  (c) 18  (d) 21  (e) 25

- Any 9-digit number (this gives 9 possibilities).

In total, 9 + 6 + 6 = 21 such numbers.
How many positive integers less than one billion (i.e. $10^9$) are divisible by 9 and have all digits equal?

(a) 10  (b) 14  (c) 18  (d) 21  (e) 25

- Any 9-digit number (this gives 9 possibilities).
- 3-digit or 6-digit numbers: common digit is either 3, 6, or 9 ($2 \cdot 3 = 6$ possibilities).
How many positive integers less than one billion (i.e. $10^9$) are divisible by 9 and have all digits equal?

(a) 10  (b) 14  (c) 18  (d) 21  (e) 25

- Any 9-digit number (this gives 9 possibilities).
- 3-digit or 6-digit numbers: common digit is either 3, 6, or 9 ($2 \cdot 3 = 6$ possibilities).
- 1-, 2-, 4-, 5-, 7-, 8-digit number: common digit is 9 (6 more possibilities).
Problem # 20

How many positive integers less than one billion (i.e. $10^9$) are divisible by 9 and have all digits equal?

(a) 10  (b) 14  (c) 18  (d) 21  (e) 25

- Any 9-digit number (this gives 9 possibilities).
- 3-digit or 6-digit numbers: common digit is either 3, 6, or 9 ($2 \cdot 3 = 6$ possibilities).
- 1-, 2-, 4-, 5-, 7-, 8-digit number: common digit is 9 (6 more possibilities).
- In total, $9 + 6 + 6 = 21$ such numbers.
Three segments through point $P$ parallel to the sides of the triangle $XYZ$ divide the triangle into six disjoint subregions. The areas of three of them are shown in the picture. Find the area of $\triangle XYZ$.

(a) 50  (b) 97  (c) 100  (d) 121  (e) 144
Problem # 21

Three segments through point $P$ parallel to the sides of the triangle $XYZ$ divide the triangle into six disjoint subregions. The areas of three of them are shown in the picture. Find the area of $\triangle XYZ$.

(a) 50  (b) 97  (c) 100  (d) 121  (e) 144
Problem # 22

Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) $\frac{3}{16}$  (b) $\frac{4}{16}$  (c) $\frac{5}{16}$  (d) $\frac{6}{16}$  (e) $\frac{7}{16}$
Problem # 22

Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) $\frac{3}{16}$  (b) $\frac{4}{16}$  (c) $\frac{5}{16}$  (d) $\frac{6}{16}$  (e) $\frac{7}{16}$

- Label the points with numbers from 1 to 6.
Problem # 22

Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) 3/16  (b) 4/16  (c) 5/16  (d) 6/16  (e) 7/16

- Label the points with numbers from 1 to 6.
- Let $A_j$ be the event that all six points can be covered by some arc of length 1 and point number $j$ is the first one in this arc if one counts clockwise.
Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) $\frac{3}{16}$  
(b) $\frac{4}{16}$  
(c) $\frac{5}{16}$  
(d) $\frac{6}{16}$  
(e) $\frac{7}{16}$

- Label the points with numbers from 1 to 6.
- Let $A_j$ be the event that all six points can be covered by some arc of length 1 and point number $j$ is the first one in this arc if one counts clockwise.
- $A_j \iff$ remaining 5 points fall onto the semicircle of length 1 which “starts” at the $j^{th}$ point.
Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) $\frac{3}{16}$  (b) $\frac{4}{16}$  (c) $\frac{5}{16}$  (d) $\frac{6}{16}$  (e) $\frac{7}{16}$

- Label the points with numbers from 1 to 6.
- Let $A_j$ be the event that all six points can be covered by some arc of length 1 and point number $j$ is the first one in this arc if one counts clockwise.
- $A_j \iff$ remaining 5 points fall onto the semicircle of length 1 which “starts” at the $j^{th}$ point.
- The probability of each point being placed onto this semicircle is $\frac{1}{2}$. 
Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) 3/16  (b) 4/16  (c) 5/16  (d) 6/16  (e) 7/16

- Label the points with numbers from 1 to 6.
- Let $A_j$ be the event that all six points can be covered by some arc of length 1 and point number $j$ is the first one in this arc if one counts clockwise.
- $A_j \iff$ remaining 5 points fall onto the semicircle of length 1 which “starts” at the $j^{th}$ point.
- The probability of each point being placed onto this semicircle is $\frac{1}{2}$.
- The probability of $A_j$ is therefore $\frac{1}{2^5} = \frac{1}{32}$. 
Problem # 22

Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) 3/16  (b) 4/16  (c) 5/16  (d) 6/16  (e) 7/16

- Label the points with numbers from 1 to 6.
- Let $A_j$ be the event that all six points can be covered by some arc of length 1 and point number $j$ is the first one in this arc if one counts clockwise.
- $A_j \iff$ remaining 5 points fall onto the semicircle of length 1 which “starts” at the $j^{th}$ point.
- The probability of each point being placed onto this semicircle is $\frac{1}{2}$.
- The probability of $A_j$ is therefore $\frac{1}{2^5} = \frac{1}{32}$.
- $A_j$’s are mutually exclusive.
Suppose 6 points are placed at random on a circle of circumference 2. What is the probability that all six can be covered by some arc of length 1?

(a) $\frac{3}{16}$  (b) $\frac{4}{16}$  (c) $\frac{5}{16}$  (d) $\frac{6}{16}$  (e) $\frac{7}{16}$

- Label the points with numbers from 1 to 6.
- Let $A_j$ be the event that all six points can be covered by some arc of length 1 and point number $j$ is the first one in this arc if one counts clockwise.
- $A_j \iff$ remaining 5 points fall onto the semicircle of length 1 which “starts” at the $j^{th}$ point.
- The probability of each point being placed onto this semicircle is $\frac{1}{2}$.
- The probability of $A_j$ is therefore $\frac{1}{2^5} = \frac{1}{32}$.
- $A_j$’s are mutually exclusive.
- $\sum_{j=1}^{6} \mathbb{P}(A_j) = \frac{6}{32} = \frac{3}{16}$. 
In the figure below, three congruent circles are tangent to each other and to the sides of an equilateral triangle of side length $a$ as shown. What is the radius of the circles?

(a) $\frac{a}{4}$  
(b) $\frac{a}{2\sqrt{3}}$  
(c) $\frac{\sqrt{3}-1}{4}a$  
(d) $\frac{\sqrt{3}}{9}a$  
(e) $\frac{2}{7}a$
In the figure below, three congruent circles are tangent to each other and to the sides of an equilateral triangle of side length $a$ as shown. What is the radius of the circles?

(a) $\frac{a}{4}$  
(b) $\frac{a}{2\sqrt{3}}$  
(c) $\frac{\sqrt{3}-1}{4}a$  
(d) $\frac{\sqrt{3}}{9}a$  
(e) $\frac{2}{7}a$
Simplify the expression (where $a$, $b$, and $c$ are different real numbers)

\[
\frac{(x - a)(x - b)}{(c - a)(c - b)} + \frac{(x - b)(x - c)}{(a - b)(a - c)} + \frac{(x - c)(x - a)}{(b - c)(b - a)}.
\]

(a) 0 (b) $x^2 - (a + b)x + ab + 1$ (c) $\frac{(x-a)(x-b)(x-c)}{(a-b)(b-c)(c-a)}$ (d) $\frac{3x^2}{(a-b)(b-c)(c-a)}$ (e) 1
Simplify the expression (where \(a\), \(b\), and \(c\) are different real numbers)

\[
\frac{(x - a)(x - b)}{(c - a)(c - b)} + \frac{(x - b)(x - c)}{(a - b)(a - c)} + \frac{(x - c)(x - a)}{(b - c)(b - a)}.
\]

(a) 0  \hspace{1cm} (b) \(x^2 - (a + b)x + ab + 1\)  \hspace{1cm} (c) \(\frac{(x-a)(x-b)(x-c)}{(a-b)(b-c)(c-a)}\)

(d) \(\frac{3x^2}{(a-b)(b-c)(c-a)}\)  \hspace{1cm} (e) 1

- This expression is a quadratic polynomial of \(x\).
Problem # 24

Simplify the expression (where \( a, b, \) and \( c \) are different real numbers)

\[
\frac{(x - a)(x - b)}{(c - a)(c - b)} + \frac{(x - b)(x - c)}{(a - b)(a - c)} + \frac{(x - c)(x - a)}{(b - c)(b - a)}.
\]

(a) 0 \hspace{1cm} (b) \( x^2 - (a + b)x + ab + 1 \) \hspace{1cm} (c) \( \frac{(x-a)(x-b)(x-c)}{(a-b)(b-c)(c-a)} \)

(d) \( \frac{3x^2}{(a-b)(b-c)(c-a)} \) \hspace{1cm} (e) 1

- This expression is a quadratic polynomial of \( x \).
- It equals 1 at three different points: \( x = a, b, \) and \( c \).
Simplify the expression (where \(a\), \(b\), and \(c\) are different real numbers)

\[
\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)}.
\]

(a) 0 (b) \(x^2 - (a + b)x + ab + 1\) (c) \(\frac{(x-a)(x-b)(x-c)}{(a-b)(b-c)(c-a)}\)

(d) \(\frac{3x^2}{(a-b)(b-c)(c-a)}\) (e) 1

- This expression is a quadratic polynomial of \(x\).
- It equals 1 at three different points: \(x = a, b,\) and \(c\).
- This can happen only if this quadratic polynomial is identically equal to 1.
\( \triangle ABC \) is an equilateral triangle of unit area. Points \( P, Q, \) and \( R \) are chosen on the sides \( \overline{AB}, \overline{BC}, \) and \( \overline{CA} \) so that \( \frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RA} = \frac{2}{1} \). Find the area of the triangle \( \triangle XYZ \).

\( \phantom{\quad} \)

(a) \( \frac{1}{4} \)  
(b) \( \frac{1}{7} \)  
(c) \( \frac{1}{10} \)  
(d) \( \frac{2}{3\sqrt{3}} \)  
(e) \( \frac{\sqrt{3}}{8} \)
\( \triangle ABC \) is an equilateral triangle of unit area. Points \( P, Q, \) and \( R \) are chosen on the sides \( \overline{AB}, \overline{BC}, \) and \( \overline{CA} \) so that \( \frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RA} = \frac{2}{1}. \) Find the area of the triangle \( \triangle XYZ. \)

(a) \( \frac{1}{4} \)  
(b) \( \frac{1}{7} \)  
(c) \( \frac{1}{10} \)  
(d) \( \frac{2}{3\sqrt{3}} \)  
(e) \( \frac{\sqrt{3}}{8} \)
Problem # 26

What is the last digit of the sum
1! + 2! + 3! + ... + 2010! + 2011! ?

(a) 0  (b) 4  (c) 8  (d) 3  (e) 7
Problem # 26

What is the last digit of the sum
$1! + 2! + 3! + \ldots + 2010! + 2011!$ ?

(a) 0  (b) 4  (c) 8  (d) 3  (e) 7

- For $n \geq 5$ the number $n!$ is divisible by both 2 and 5.
What is the last digit of the sum $1! + 2! + 3! + \ldots + 2010! + 2011!$?

(a) 0  (b) 4  (c) 8  (d) 3  (e) 7

- For $n \geq 5$ the number $n!$ is divisible by both 2 and 5.
- Hence it is divisible by ten and ends with a 0.
Problem # 26

What is the last digit of the sum
\[1! + 2! + 3! + \ldots + 2010! + 2011! \, ?\]

(a) 0  (b) 4  (c) 8  (d) 3  (e) 7

- For \( n \geq 5 \) the number \( n! \) is divisible by both 2 and 5.
- Hence it is divisible by ten and ends with a 0.
- Therefore, we only need to find the last digit of
  \[1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33,\]
  which is 3.
Problem # 27

How many perfect squares divide the number $4! \cdot 5! \cdot 6!$?

(a) 22 (b) 10 (c) 120 (d) 36 (e) 45
Problem # 27

How many perfect squares divide the number $4! \cdot 5! \cdot 6!$ ?

(a) 22  (b) 10  (c) 120  (d) 36  (e) 45

\[ 4! \cdot 5! \cdot 6! = (2 \cdot 3 \cdot 2^2) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3)) = 2^{10} \cdot 3^4 \cdot 5^2. \]
How many perfect squares divide the number $4! \cdot 5! \cdot 6!$?

(a) 22  (b) 10  (c) 120  (d) 36  (e) 45

- $4! \cdot 5! \cdot 6! = (2 \cdot 3 \cdot 2^2)(2 \cdot 3 \cdot 2^2 \cdot 5)(2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3)) = 2^{10} \cdot 3^4 \cdot 5^2$.

- The prime factorization of a perfect square dividing $4! \cdot 5! \cdot 6!$ has to consist of even powers of the primes 2, 3, and 5.
How many perfect squares divide the number $4! \cdot 5! \cdot 6!$ ?

(a) 22 \hspace{1cm} (b) 10 \hspace{1cm} (c) 120 \hspace{1cm} (d) 36 \hspace{1cm} (e) 45

- $4! \cdot 5! \cdot 6! = (2 \cdot 3 \cdot 2^2) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3)) = 2^{10} \cdot 3^4 \cdot 5^2$.
- The prime factorization of a perfect square dividing $4! \cdot 5! \cdot 6!$ has to consist of even powers of the primes 2, 3, and 5.
- There are 6 choices for the exponent in the power of 2 (0, 2, 4, 6, 8, 10),
How many perfect squares divide the number $4! \cdot 5! \cdot 6!$?

(a) 22  (b) 10  (c) 120  (d) 36  (e) 45

- $4! \cdot 5! \cdot 6! = (2 \cdot 3 \cdot 2^2) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3)) = 2^{10} \cdot 3^4 \cdot 5^2$.
- The prime factorization of a perfect square dividing $4! \cdot 5! \cdot 6!$ has to consist of even powers of the primes 2, 3, and 5.
- There are 6 choices for the exponent in the power of 2 (0, 2, 4, 6, 8, 10),
- 3 choices for the power of 3 (0, 2, 4),
How many perfect squares divide the number $4! \cdot 5! \cdot 6!$?

(a) 22 (b) 10 (c) 120 (d) 36 (e) 45

- $4! \cdot 5! \cdot 6! = (2 \cdot 3 \cdot 2^2) \cdot (2 \cdot 3^2 \cdot 5) \cdot (2 \cdot 3^2 \cdot 5 \cdot (2 \cdot 3)) = 2^{10} \cdot 3^4 \cdot 5^2$.

- The prime factorization of a perfect square dividing $4! \cdot 5! \cdot 6!$ has to consist of even powers of the primes 2, 3, and 5.

- There are 6 choices for the exponent in the power of 2 (0, 2, 4, 6, 8, 10),

- 3 choices for the power of 3 (0, 2, 4),

- two choices for the power of 5 (0, 2).
How many perfect squares divide the number $4! \cdot 5! \cdot 6$?

(a) 22 (b) 10 (c) 120 (d) 36 (e) 45

- $4! \cdot 5! \cdot 6! = (2 \cdot 3 \cdot 2^2) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5) \cdot (2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3)) = 2^{10} \cdot 3^4 \cdot 5^2$.
- The prime factorization of a perfect square dividing $4! \cdot 5! \cdot 6!$ has to consist of even powers of the primes 2, 3, and 5.
- There are 6 choices for the exponent in the power of 2 (0, 2, 4, 6, 8, 10),
- 3 choices for the power of 3 (0, 2, 4),
- two choices for the power of 5 (0, 2).
- The total number of possibilities is then $6 \cdot 3 \cdot 2 = 36$. 
Let $O$ be a point inside the square $ABCD$ such that its distances to the vertices are $OA = 10$, $OB = 9$, $OC = 5$, $OD = x$. What is the value of $x$?

(a) $\sqrt{6}$  
(b) 6  
(c) 7  
(d) $\sqrt{44}$  
(e) $\sqrt{156}$
Let $O$ be a point inside the square $ABCD$ such that its distances to the vertices are $OA = 10$, $OB = 9$, $OC = 5$, $OD = x$. What is the value of $x$?

(a) $\sqrt{6}$  (b) 6  (c) 7  (d) $\sqrt{44}$  (e) $\sqrt{156}$

- $OA^2 = a^2 + d^2$, $OB^2 = a^2 + b^2$, $OC^2 = b^2 + c^2$, and $OD^2 = c^2 + d^2$. 

$$10^2 + 5^2 = 9^2 + x^2 \Rightarrow x = \sqrt{100 + 25 - 81} = \sqrt{44}.$$
Let $O$ be a point inside the square $ABCD$ such that its distances to the vertices are $OA = 10$, $OB = 9$, $OC = 5$, $OD = x$. What is the value of $x$?

(a) $\sqrt{6}$  (b) 6  (c) 7  (d) $\sqrt{44}$  (e) $\sqrt{156}$

- $OA^2 = a^2 + d^2$, $OB^2 = a^2 + b^2$, $OC^2 = b^2 + c^2$, and $OD^2 = c^2 + d^2$.
- $OA^2 + OC^2 = OB^2 + OD^2 = a^2 + b^2 + c^2 + d^2$. 
Let $O$ be a point inside the square $ABCD$ such that its distances to the vertices are $OA = 10$, $OB = 9$, $OC = 5$, $OD = x$. What is the value of $x$?

(a) $\sqrt{6}$  (b) 6  (c) 7  (d) $\sqrt{44}$  (e) $\sqrt{156}$

- $OA^2 = a^2 + d^2$, $OB^2 = a^2 + b^2$, $OC^2 = b^2 + c^2$, and $OD^2 = c^2 + d^2$.
- $OA^2 + OC^2 = OB^2 + OD^2 = a^2 + b^2 + c^2 + d^2$.
- $10^2 + 5^2 = 9^2 + x^2 \implies x = \sqrt{100 + 25 - 81} = \sqrt{44}$. 
Each of the seven vertices of a regular heptagon is colored garnet or black. Two colorings are considered the same if one coloring is a rotation of the other. How many different colorings are possible?

(a) 18  (b) 20  (c) 22  (d) 24  (e) 30
Problem # 29

Each of the seven vertices of a regular heptagon is colored garnet or black. Two colorings are considered the same if one coloring is a rotation of the other. How many different colorings are possible?

(a) 18  (b) 20  (c) 22  (d) 24  (e) 30

- All together there are $2^7$ ways to color the vertices garnet or black – we call them “patterns”.
Each of the seven vertices of a regular heptagon is colored garnet or black. Two colorings are considered the same if one coloring is a rotation of the other. How many different colorings are possible?

(a) 18    (b) 20    (c) 22    (d) 24    (e) 30

- All together there are $2^7$ ways to color the vertices garnet or black – we call them “patterns”.
- If the coloring is non-trivial (not all black or all garnet), then a rotation of a pattern produces a different pattern (because 7 is a prime number).
Problem # 29

Each of the seven vertices of a regular heptagon is colored garnet or black. Two colorings are considered the same if one coloring is a rotation of the other. How many different colorings are possible?

(a) 18  (b) 20  (c) 22  (d) 24  (e) 30

- All together there are $2^7$ ways to color the vertices garnet or black – we call them “patterns”.
- If the coloring is non-trivial (not all black or all garnet), then a rotation of a pattern produces a different pattern (because 7 is a prime number).
- So each non-trivial coloring corresponds to 7 non-trivial patterns. $\Rightarrow \frac{2^7 - 2}{7} = 18$ colorings.
Each of the seven vertices of a regular heptagon is colored garnet or black. Two colorings are considered the same if one coloring is a rotation of the other. How many different colorings are possible?

(a) 18  (b) 20  (c) 22  (d) 24  (e) 30

- All together there are $2^7$ ways to color the vertices garnet or black – we call them “patterns”.
- If the coloring is non-trivial (not all black or all garnet), then a rotation of a pattern produces a different pattern (because 7 is a prime number).
- So each non-trivial coloring corresponds to 7 non-trivial patterns. \[ \Rightarrow \frac{2^7 - 2}{7} = 18 \] colorings.
- +2 trivial colorings (all black or all garnet). \[ \Rightarrow \text{total number } 18 + 2 = 20. \]
Problem # 30

In the figure below the large circle has radius 1 and center $O$. $\overline{PQ}$ and $\overline{RS}$ are its diameters and are perpendicular. $\overline{OP}$, $\overline{OR}$, $\overline{OQ}$, and $\overline{OS}$ are diameters of the four small circles below. The point $T$ is a point of intersection of two of the small circles. Let $a_1$ be the area of the shaded region between the arcs $\widehat{TS}$, $\widehat{TQ}$, and $\widehat{QS}$, and $a_2$ be the area of the shaded region between the two arcs $\widehat{OT}$. What is the value of $a_1 - a_2$?
In the figure below the large circle has radius 1 and center $O$. $\overline{PQ}$ and $\overline{RS}$ are its diameters and are perpendicular. $\overline{OP}$, $\overline{OR}$, $\overline{OQ}$, and $\overline{OS}$ are diameters of the four small circles below. The point $T$ is a point of intersection of two of the small circles. Let $a_1$ be the area of the shaded region between the arcs $\widehat{TS}$, $\widehat{TQ}$, and $\widehat{QS}$, and $a_2$ be the area of the shaded region between the two arcs $\widehat{OT}$. What is the value of $a_1 - a_2$?