1. (e) You can manage this without our help.

2. (c) Clearly there are \( \leq 8 \) quarters. Since there are more quarters than dimes, there are \( \leq 7 \) dimes. Thus, the dimes contribute at most 70 cents to the total so that the quarters contribute at least \( 215 - 70 = 145 \) cents to the total. As 215 ends with the digit 5, we also see that there are an odd number of quarters. It follows that there must be 7 quarters and, hence, 4 dimes.

3. (b) Note that \( 77/165 = 7/15 \). Simplifying the equation in the problem, we have \( 4x/3 = 1 \), so \( x = 3/4 = 0.75 \).

4. (b) The ratio of the area of the sector \( OAB \) to the area of the sector \( ODC \) is the square of the ratio of the length of \( OA \) to the length of \( OD \). Hence, the area of sector \( OAB \) is 9 times as big as the area of sector \( ODC \). Thus, the area of the region \( ABCD \) is 8 times as big as the area of sector \( ODC \).

5. (b) Let \( b \) be the number of boys that took the test and \( g \) the number of girls. We are given that \( 2b/3 = 3g/4 \), so \( b = 9g/8 \). So the desired ratio is

\[
\frac{(2b/3) + (3g/4)}{b + g} = \frac{2 \cdot (3g/4)}{9g/8 + g} = \frac{3/2}{17/8} = \frac{12}{17}.
\]

6. (c) In the left-most column, the digit 5 must be placed in the bottom square. Then there is only one possibility for where the digit 3 can be placed in the bottom row, namely in the bottom right corner.

7. (a) Let \( N = 19700019d \). Clearly, \( d \) cannot be even or 5. Also, if \( d \) is a multiple of 3, then \( N \) is divisible by 3 which can be checked by summing its digits. As 197 is an apparent divisor of \( N \) if \( d = 7 \), we are left with only the possibility that \( d = 1 \).

8. (d) The altitudes drawn from \( A \) in triangles \( \triangle ABD \) and \( \triangle ACD \) are the same. The product of the areas is maximized when the product of the lengths of the bases, that is \( BD \cdot CD \), is maximized. Letting \( x = BD \), we have \( 4 - x = CD \). Thus, we want to know where \( x(4 - x) = 4 - (x - 2)^2 \) is maximized. As \( 4 - (x - 2)^2 \) cannot be greater than 4 and the value of 4 is attained at \( x = 2 \), we deduce that that we want \( x = 2 \). One checks that \( S_1 = S_2 = 2\sqrt{3} \) and the answer follows.

9. (d) Let \( \log x \) denote the logarithm of \( x \) to any fixed base that you want. Then using the change of base formula and simplifying, we obtain

\[
\frac{\log_2 3 \cdot \log_4 5 \cdot \log_6 7}{\log_4 3 \cdot \log_6 5 \cdot \log_8 7} = \frac{\log 3}{\log 4} \cdot \frac{\log 5}{\log 6} \cdot \frac{\log 7}{\log 8} = \frac{\log 8}{\log 2} = \frac{3\log 2}{\log 2} = 3.
\]

10. (e) If \( r(x) \) is the remainder, then there is a quotient \( q(x) \) such that

\[
x^{2006} - x^{2005} + (x + 1)^2 = (x^2 - 1)q(x) + r(x).
\]

Plugging in \( x = \pm 1 \), we see that \( r(1) = 4 \) and \( r(-1) = 2 \). The answer follows.

11. (c) Let \( r \) denote the radius of the smaller circle and \( R \) the radius of the larger circle, as shown. Then the Pythagorean Theorem implies that \( R^2 - r^2 = 36 \). Hence, the area of the shaded region is \( \pi R^2 - \pi r^2 = 36\pi \).
12. **(b)** Observe that $86 = 3b^2 + 2b + 1$ implies $b(3b + 2) = 85 = 5 \cdot 17$. Hence, $b = 5$, and the answer is $b^2 + 2b + 3 = 25 + 10 + 3 = 38$.

Alternatively, the sum of the base $b$ numbers 321 and 123 is $(3b^2 + 2b + 1) + (b^2 + 2b + 3) = 4(b^2 + b + 1)$. Note that $b^2 + b = b(b + 1)$ is the product of two consecutive integers and, hence, even. Thus, $b^2 + b + 1$ is odd. In base 10, this means that the sum of 86 and the number we want must be divisible by 4 and not 8. The only choice that satisfies this is 38.

13. **(b)** The 5-digit number must end with 25 or 75 (on the right). The other digits can be arbitrary but must be distinct from each other, the two right-most digits and zero. This leaves $7 \cdot 6 \cdot 5 = 210$ choices for the remaining digits. The answer is $2 \cdot 210 = 420$.

14. **(e)** Setting $x = 2$ and then $x = -1$ gives $3f(2) + 2f(-1) = 13$ and $3f(-1) + 2f(2) = 7$. Multiplying through in the first equation by 3 and in the second equation by 2 and then subtracting gives $5f(2) = 25$. Hence, $f(2) = 5$.

15. **(d)** By the given information and the Pythagorean Theorem,

$$a^2 + b^2 = c^2 = (49 - a)^2 = 49^2 - 98a + a^2$$

so that $b^2 = 49(49 - 2a)$. We deduce that $49 - 2a$ is a square. As it is also odd, we must have $49 - 2a \in \{1, 9, 25\}$ so that $a \in \{24, 20, 12\}$. Clearly, $a = 24$, $a < b < c$ and $a + c = 49$ are not consistent. If $a = 20$, then $b = 21$. If $a = 12$, then $b = 35$. In either case, the area of the triangle is $ab/2 = 210$.

16. **(a)** Let $f(x) = x^4 + 8x^3 - 40x + 125$. The Rational Root Test implies that the rational roots of $f(x)$ must be integers that divide 125, that is they are in the set $S = \{\pm 1, \pm 5, \pm 25, \pm 125\}$. If $a \in S$, then $a^2$ is odd so that $a^2 - 1$ and $a^2 + 1$ are consecutive even integers. It follows that one of them is divisible by 4 so that $a^4 - 1 = (a^2 - 1)(a^2 + 1)$ is divisible by 8. If $f(a) = 0$, then

$$0 = a^4 + 8a^3 - 40a + 125 = (a^4 - 1) + 8a^3 - 40a + 126.$$  

But this is impossible as $a^4 - 1 + 8a^3 - 40a$ is the sum of three expressions each of which is divisible by 8 and 126 is not divisible by 8. In other words, the above equation implies that $(a^4 - 1) + 8a^3 - 40a = -126$, but the left-hand side of this equation is an integer divisible by 8 and the right-hand side is not. It follows that we cannot have $f(a) = 0$ for each $a \in S$, giving the desired result.

Alternatively, it is easy to see that 1 and $-1$ are not roots of $f(x)$. Also, $f(\pm 5)$ is a sum of four terms exactly three of which are divisible by 125 and so $f(\pm 5) \neq 0$ (this is for reasons similar to how we argued that $f(a) \neq 0$ above). The number $f(\pm 25)$ is a sum of the number $25^4 + 8 \cdot 25^3$, which is divisible by $5^4$, and the number $\mp 40 \cdot 25 + 125 = (\mp 8 + 1) \cdot 125$, which is not divisible by $5^4$. It follows that $f(\pm 25) \neq 0$. Finally, $f(\pm 125)$ is also not 0 as it is a sum of three terms divisible by $5^4$ and one that is not.

17. **(d)** The area of one semi-circle is equal to the area of two “petals” and one of the four congruent regions in the square that is outside the petals. It follows that the answer is equal to the area of four semi-circles minus the area of the square, that is $2 \cdot \pi (1/2)^2 - 1 = (\pi - 2)/2$. As $\pi$ is between 3.1 and 3.2, the exact answer is greater than 0.55 and less than 0.6.

18. **(a)** As $a$, $b$ and $c$ are positive, the number $c + a + b$ is larger than the absolute value of $c - a - b$. From

$$(c + a + b)(c - a - b) = c^2 - (a + b)^2 = c^2 - a^2 - b^2 - 2ab = 101 - 2 \cdot 72 = -43,$$

we deduce that $c + a + b = 43$ and $c - a - b = -1$.

19. **(e)** If one box has exactly $a$ white marbles and the other box has exactly $b$ white marbles, then we deduce that $(a/20)(b/20) = 0.21$ so that $ab = 0.21 \cdot 400 = 84$. As $a$ and $b$ are integers that are $\leq 20$, we deduce that the set
\( \{a, b\} \) is either \( \{6, 14\} \) or \( \{7, 12\} \). Since the total number of black marbles is different from the total number of white marbles, we know \( a + b \neq 20 \). Thus, the probability that both marbles drawn are black is \( \frac{(20 - a)(20 - b)}{400} = \frac{(20 - 7)(20 - 12)}{400} = 13 \cdot \frac{8}{400} = 0.26 \).

20. (c) We want to know that number of positive \( x \) for which either \( \sqrt{x} = x^4 - 1 \) or \( -\sqrt{x} = x^4 - 1 \). The graph of \( y = x^4 - 1 \) intersects the \( y \)-axis at \( (0, -1) \) and the value of \( y \) increases as \( x \) increases. As \( x^4 \) grows much quicker than \( \sqrt{x} \) as \( x \) increases, it is not difficult to see that the graph of \( y = x^4 - 1 \) intersects the graph of \( y = -\sqrt{x} \) exactly once, and then, at a larger value of \( x \), it intersects the graph of \( y = \sqrt{x} \) exactly once. Hence, there are exactly two solutions to the given equation.

Alternatively, setting \( t = \sqrt{x} \), we want to know how many positive values of \( t \) satisfy either \( t = t^8 - 1 \) or \( -t = t^8 - 1 \). On the other hand, Descartes’ Rule of Signs implies that each of \( t^8 - t - 1 \) and \( t^8 + t - 1 \) has at most 1 positive root and has an odd number of positive roots. For positive \( t \), we cannot have \( t^8 - t - 1 = t^8 + t - 1 \). Hence, there are exactly two positive \( t \) satisfying either \( t = t^8 - 1 \) or \( -t = t^8 - 1 \).

21. (a) If \( \alpha \) is a root of \( f(x) = x^3 + (a - 1)x^2 - ax + 1 \) and \( g(x) = x^2 + ax + 1 \), then it is a root of \( h(x) = f(x) - g(x)(x - 1) \) since \( h(\alpha) = f(\alpha) - g(\alpha)(\alpha - 1) = 0 - 0 \cdot (\alpha - 1) = 0 \). Since \( h(x) = -x + 2 \), we deduce \( \alpha = 2 \). Now, \( g(\alpha) = 0 \) implies that \( g(2) = 0 \) so that \( 2a + 5 = 0 \) and \( a = -5/2 \).

22. (d) We refer to the figure to the right. The coordinates of \( C \) can be obtained by adding the \( x \)-coordinates of \( B \) and \( D \) and adding the \( y \)-coordinates of \( B \) and \( D \). Thus, \( C \) is the point \((30, y + 10)\). One can compute the area of the parallelogram as the area of rectangle \( XAYC \) minus the sum of the areas of the triangles \( \triangle YCB \), \( \triangle YCB \), \( \triangle CXD \) and \( \triangle XAD \). We deduce that the area of the parallelogram is

\[
30(y + 10) - (1/2)(30 - 10 + 10(y + 10) + 30 - 10 + 10(y + 10)) = 20y - 100.
\]

Since the area is 600, we obtain \( y = 35 \).

Alternatively, the vectors \( \overrightarrow{AB} = (20, 10) \) and \( \overrightarrow{AD} = (10, y) \) are vectors lying along adjacent edges of the parallelogram. The area of the parallelogram can be computed by computing the absolute value of the determinant of the matrix with rows formed from the components of these vectors. In other words,

\[
600 = \det \begin{pmatrix} 20 & 10 \\ 10 & y \end{pmatrix} = 20y - 100,
\]

which again implies \( y = 35 \). (Note that \( 20y - 100 = -600 \) would lead to \( y < 0 \), contrary to the figure shown.)

23. (d) Since the remainder is 6, we have \( n > 6 \). If \( q \) is the quotient, we have \( 2006 = nq + 6 \) so that \( nq = 2000 \). Thus, \( n \) is a divisor of 2000 that is \( > 6 \). It is not hard to see that this is both necessary and sufficient for \( n \) to satisfy the conditions in the problem. As \( 2000 = 2^4 \cdot 5^3 \), it has \((4 + 1)(3 + 1) = 20 \) positive integer divisors. This includes the divisors 1, 2, 4 and 5 that are \( \leq 6 \). Hence, the answer is 20 - 4 = 16.

24. (c) Since \( x^2 + ax + b = (x - \sin 15^\circ)(x - \cos 15^\circ) \), we deduce that \( a = -\sin 15^\circ \cdot \cos 15^\circ \) and \( b = \sin 15^\circ \cdot \cos 15^\circ \). Observe that

\[
a^2 = \sin^2 15^\circ + 2 \sin 15^\circ \cos 15^\circ + \cos^2 15^\circ = 1 + \sin 30^\circ = \frac{3}{2} \quad \text{and} \quad b = \frac{1}{2} \cdot 2 \sin 15^\circ \cos 15^\circ = \frac{1}{2} \cdot \sin 30^\circ = \frac{1}{4}.
\]

We deduce \( a^4 - b^2 = (3/2)^2 - (1/4)^2 = 35/16 \).

25. (e) By definition, \( s_8 = 2006 \). As \( 2^{10} < 2006 < 2^{11} \), we get that \( s_3 \) is in the interval \((10, 11)\). As \( 2^3 < 10 < 11 < 2^4 \), we obtain \( s_4 \in (3, 4) \). Similarly, \( s_5 \in (1, 2) \) and then \( s_6 < 1 \).
26. (d) In each of the 9 codes, one of the digits $a$, $b$, $c$ and $d$ appears in a correct position. As there are exactly 4 possibilities for a correct digit in a correct position, there must be at least 3 codes that contain one of $a$, $b$, $c$ and $d$ in a correct position. (If the last sentence is clear, great. Otherwise, you might want to look up the Pigeonhole Principle.) As 2 is the only digit that occurs 3 times in the same position and this happens in the third position, we deduce that $c = 2$. Of the 6 codes that do not have 2 in the third position, there are no more than two occurrences of the same digit in the same position. It follows that each of $a$, $b$ and $d$ must occur in its correct position exactly twice among these 6 codes. In particular, this gives $d = 6$.

27. (a) Let $f(x) = x^5 - 6x^4 + Ax^3 + Bx^2 + Cx + D$. The sum of the roots (counted to their multiplicity) of a polynomial of degree $n$ is minus the coefficient of $x^{n-1}$ divided by the coefficient of $x^n$. Hence, the sum of the roots of $f(x)$ is 6. Note that $f(x)$ has exactly 5 complex roots (counting again a root to its multiplicity). As the coefficients are integers and, hence, real, the imaginary roots come in conjugate pairs. Thus, $1 + i$, $1 - i$, $1 + 2i$ and $1 - 2i$ are all roots of $f(x)$. The sum of the roots of $f(x)$ being 6 implies that the fifth root of $f(x)$ is 2. As $f(x)$ has a leading coefficient of 1, we obtain

$$f(x) = (x - 2)(x - 1 - i)(x - 1 + i)(x - 1 - 2i)(x - 1 + 2i).$$

Thus,

$$-5 + A + B + C + D = f(1) = -(-i) \cdot i \cdot (2i) = -4.$$

Therefore, $A + B + C + D = 5 - 4 = 1$.

28. (c) Setting $t = x^4 - 4x^2 + 4 = (x^2 - 2)^2$, we see that $t \geq 0$ and, from the given inequality, that $|t - 10| \geq |t + 10|$. This inequality is equivalent to the assertion that the distance from $t$ to 10 on the number line is greater than or equal to the distance from $t$ to $-10$. As $t \geq 0$, this happens precisely if $t = 0$. Thus, $(x^2 - 2)^2 = 0$, and we obtain $x = \pm \sqrt{2}$.

29. (c) Using the trigonometric identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$ with $A = 2\theta$ and $B = \theta$ followed by the identities $\cos(2\theta) = 2\cos^2 \theta - 1$, $\sin(2\theta) = 2\sin \theta \cos \theta$ and $\sin^2 \theta = 1 - \cos^2 \theta$ gives

$$\cos(3\theta) = \cos(2\theta) \cos \theta - \sin(2\theta) \sin \theta = 2\cos^3 \theta - \cos \theta - 2\sin^2 \theta \cos \theta = 2\cos^3 \theta - \cos \theta - 2(1 - \cos^2 \theta) \cos \theta = 4\cos^3 \theta - 3\cos \theta.$$

Hence, $f(\theta)$, which is only defined when $\cos \theta \neq 0$, can be rewritten as

$$f(\theta) = \cos \theta + 4\cos^2 \theta - 3 = \left(2\cos \theta + \frac{1}{4}\right)^2 - \frac{49}{16}.$$ 

As $f(\theta)$ is a nonnegative number plus $-49/16$, we deduce $f(\theta) \geq -49/16$. On the other hand, there is a $\theta$ such that $\cos \theta = -1/8$ and for this value of $\theta$, the above expression for $f(\theta)$ implies $f(\theta) = -49/16$. Thus, the least value of $f(\theta)$ as $\theta$ varies over its domain is $-49/16$. 

30. (b) Both $m$ and the sum of the digits of $m$ have the same remainder when divided by 9. In other words, $s(m)$ and $m$ have the same remainder when divided by 9. By replacing $m$ with $s(m)$ and then $s(s(m))$, we deduce that the numbers $s(s(s(m)))$, $s(s(m))$, $s(m)$ and $m$ all have the same remainder when divided by 9. Observe that

$$2^{2006} = 4 \cdot (2^{2004} - 1) + 4 = 4 \cdot (2^6 - 1)(2^{1998} + 2^{1992} + \ldots + 2^6 + 1) + 4.$$

As $2^6 - 1 = 63$ is divisible by 9, this last expression can be written as $9q + 4$ where $q$ is some integer. This means that the remainder when $m = 2^{2006}$ is divided by 9 is 4. Hence, we want an answer that has a remainder of 4 when we divide by 9. This eliminates two of our choices. We justify the answer is (b) by showing that $s(s(s(m)))$, with $m = 2^{2006}$, is $< 13$. As $2^3 < 10$, we have $2^{2006} < (2^3)^{700} < 10^{700}$. Hence, $m$ has at most 700 digits each at most 9. Therefore, $s(m) \leq 700 \cdot 9 = 6300$. The sum of the digits of a positive integer $\leq 6300$ is maximized if the positive integer is 5999. Hence, $s(s(m)) \leq 32$. The sum of the digits of a positive integer $\leq 32$ is maximized if the positive integer is 29. So finally we obtain $s(s(s(m))) \leq 11$, and the answer follows.