High School Math Contest  
University of South Carolina  
January 30, 2010  

Notation

- When \( n \) is a positive integer, \( n! \) denotes the product \( 1 \cdot 2 \cdot 3 \cdots n \). For example, \( 3! = 1 \cdot 2 \cdot 3 = 6 \).
- We denote by \( \{1, 2, 3, \ldots, 99, 100\} \) the set of all positive integers from 1 to 100.
- We denote by \( AB \) the line segment with endpoints \( A \) and \( B \), and we denote by \( AB \) the length of the line segment \( AB \).

1. What is the least positive integer \( n \) such that \( 1 + 2 + \cdots + n > 100 \)?
   (a) 10  
   (b) 12  
   (c) 13  
   (d) 14  
   (e) 15

2. A turtle starts moving north at 4 ft/min from a point \( P \) in South Carolina. Five minutes later, a snail starts moving south at 2 ft/min from a point 50 ft due east of \( P \). What is the distance (in feet) between the two animals 5 minutes after the snail starts moving?
   (a) 50 feet  
   (b) 70 feet  
   (c) \( 50\sqrt{2} \) feet  
   (d) \( 50\sqrt{3} \) feet  
   (e) 100 feet

3. Suppose that \( a \) and \( x \) are two positive real numbers for which \( \log_a x + \log_x a = 3 \). What is the value of \( (\log_a x)^2 + (\log_x a)^2 \)?
   (a) 2  
   (b) 5  
   (c) 7  
   (d) 9  
   (e) 11

4. In a group of five friends, the sums of the ages of each group of four of them are 124, 128, 130, 136, and 142. What is the age of the youngest of the friends?
   (a) 18  
   (b) 21  
   (c) 23  
   (d) 25  
   (e) 34

5. On Wednesday, four trucks drove in a line (one directly behind the other). None of the trucks passed any of the others, so the order of the trucks never changed. How many ways are there to rearrange the order of the trucks so that on the next day, no truck is directly behind a truck that it was directly behind the day before?
   (a) 4  
   (b) 8  
   (c) 11  
   (d) 12  
   (e) 16

6. For how many integers \( x \) in the set \( \{1, 2, 3, \ldots, 99, 100\} \) is \( x^3 - x^2 \) the square of an integer?
   (a) 7  
   (b) 8  
   (c) 9  
   (d) 10  
   (e) 11
7. In the diagram below, the distance between any two adjacent dots in a row or a column is one unit. What is the area of the shaded region?

![Diagram of a shaded region with vertices at (1,1), (1,3), (3,3), and (3,1).]

(a) 5 unit²  (b) 5.5 unit²  (c) 6 unit²  (d) 6.5 unit²  (e) 7 unit²

8. The surface area of a right rectangular prism (a box) is 48 square feet, and the sum of its length, width, and height is 13 feet. What is the length of the longest diagonal connecting two corners of the box?

(a) 8 feet  (b) 9 feet  (c) 10 feet  (d) 11 feet  (e) 12 feet

9. How many positive integers \( n \) have the property that the measures (in degrees) of the interior angles of a regular \( n \)-gon are integers?

(a) 4  (b) 22  (c) 30  (d) 32  (e) 64

10. Suppose that \( \log_4 x = \frac{1}{3} \). What is the value of \( \log_x 8 \)?

(a) \( \sqrt[4]{8} \)  (b) 2  (c) \( \frac{9}{2} \)  (d) 4  (e) \( \sqrt[4]{48} \)

11. Eleven teams play in a soccer tournament. Each team must play each of the other teams exactly once. If a game ends in a tie, each team gets 1 point. For the games that do not end in a tie, the winning team gets 5 points and the losing team gets 0 points.

Which of the following is a possible value for the total number of points earned by the 11 teams by the end of the tournament?

(a) 92  (b) 196  (c) 257  (d) 290  (e) none of the previous

12. Which of the numbers below equals \( \sin 15° + \cos 15° \)?

(a) 0  (b) 1  (c) \( \frac{1 + \sqrt{3}}{2} \)  (d) \( \frac{\sqrt{6}}{2} \)  (e) \( \sqrt{3} \)

13. What is the value of the sum \( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{99}{100!} \)?

(a) \( \frac{1}{2} - \frac{1}{100!} \)  (b) \( 1 - \frac{1}{100!} \)  (c) \( 1 - \frac{1}{99!} \)  (d) \( \frac{1}{2} - \frac{1}{99!} \)  (e) 1
14. How many integers between 1 and 1000 have exactly 27 positive divisors?
   (a) 0  (b) 1  (c) 2  (d) 27  (e) 28

15. What is the coefficient of $x^{50}$ in the expansion of the following product?
   \[(1 + 2x + 3x^2 + 4x^3 + \cdots + 101x^{100}) \cdot (1 + x + x^2 + x^3 + \cdots + x^{25})\]
   (a) 50  (b) 125  (c) 501  (d) 923  (e) 1001

16. Suppose that $\triangle ABC$ is isosceles with $AB = AC$. Let $D$ be a point on the side $AB$, and let $E$ be a point on the side $AC$ for which the following are true:
   - $DE$ is parallel to $BC$.
   - $\triangle ADE$ and trapezoid $BCED$ have the same area and the same perimeter.
   - $F$ is the midpoint of $BC$.

   What is $\sin(\angle BAF)$?

   \[A\]
   \[\overline{DE}\]
   \[\overline{BC}\]
   \[\overline{F}\]
   \[\overline{AB}\]
   \[\overline{AC}\]

   (a) $\sqrt{2} - 1$  (b) $\sqrt{2}/2$  (c) $\sqrt{3} - 1$  (d) $\sqrt{3}/2$  (e) $1/2$

17. What is the remainder when $100101102103104105106107108$ is divided by $999$?
   (a) 0  (b) 27  (c) 522  (d) 936  (e) 990

18. Let $x = \sqrt{7 + 2\sqrt{6}} + \sqrt{7 - 2\sqrt{6}}$. Which of the intervals listed below contains $x$?
   (a) $(3.9, 4)$  (b) $(4.8, 4.9)$  (c) $(4.9, 5)$  (d) $(5, 5.1)$  (e) $(5.2, 5.3)$

19. Consider a plane with a standard $x$-$y$ coordinate system. A straight stick lies along the positive $y$-axis with the bottom end of the stick at the point $(0, 0)$ of the coordinate system. A point $P$ on the bottom half of the stick divides the length of the stick into two parts in the ratio $1 : 2$.
   The bottom end of the stick slides away from the origin along the positive $x$-axis while the top end of the stick moves downward along the positive $y$-axis. The path traced by the point $P$ is a part of which of the following curves?

   (a) an ellipse that is not a circle  (b) a circle  (c) a straight line  (d) a hyperbola  (e) a parabola
20. Let \( R \) be the circle \( x^2 + (y + 2)^2 = 9 \). Let \( S \) be the set of all circles in the plane such that for each circle \( C \) in \( S \), we have:

- \( C \) lies in the first quadrant outside \( R \).
- \( C \) is tangent to \( R \) and to the \( x \)-axis.

On what geometric object must the centers of the circles in \( S \) lie?

(a) an ellipse that is not a circle  (b) a circle  (c) a straight line  (d) a hyperbola  (e) a parabola

21. How many polynomials \( p(x) \) satisfy both \( p(12) = 12! \) and \( xp(x - 1) = (x - 12)p(x) \)?

(a) 0  (b) 1  (c) 2  (d) 12  (e) infinitely many

22. In the diagram below, \( \overline{AD} \) bisects \( \angle BAC \), \( AC = BC \), \( \angle ABC \) has measure \( 72^\circ \), and \( CD = 1 \). Determine the length of \( \overline{BD} \).

23. The integers from 1 to 20 are divided into \( k \) disjoint subsets \( S_1, S_2, \ldots, S_k \) such that no two integers whose sum is divisible by 5 are in the same subset. For example, 3 and 17 are not in the same subset. What is the least possible value of \( k \)?

(a) 3  (b) 4  (c) 5  (d) 7  (e) 10

24. What is the largest positive integer \( y \) such that there exists a positive integer \( x \) satisfying \( \sqrt{x} + \sqrt{y} = \sqrt{525} \)?

(a) 21  (b) 84  (c) 336  (d) 441  (e) 500
25. We are given the following information about the figure below:

- \( \triangle ABC \) is a right triangle with right angle at \( A \).
- \( \triangle A'BC \) is a right triangle with right angle at \( A' \).
- \( B'A = B'C, \quad C'A = C'B, \) and \( A'B = A'C \).
- \( \angle AB'C \) and \( \angle BC'A \) both have measure 120°.

Let \( A_1 \) denote the area of \( \triangle B'AC \), and let \( A_2 \) denote the area of \( \triangle C'AB \). Which of the following is equal to the area of \( \triangle A'BC' \)?

(a) \( \sqrt{3}(A_1 + A_2) \)  
(b) \( \frac{A_1 + A_2}{2} \)  
(c) \( \frac{A_1 + A_2}{\sqrt{3}} \)  
(d) \( \frac{\sqrt{3}(A_1 + A_2)}{2} \)  
(e) none of the previous

26. Suppose that the sum of \( k \) consecutive integers is an even integer divisible by \( k \) and that the smallest of the \( k \) consecutive integers is even. Which of the following must be true about \( k \)?

(a) \( k + 3 \) is divisible by 4  
(b) \( k + 2 \) is divisible by 4  
(c) \( k + 1 \) is divisible by 4  
(d) \( k \) is divisible by 4  
(e) none of the previous choices must be true

27. Let \( a_0, a_1, \ldots, a_n, \ldots \) be the unique sequence with the following properties:

\[
\begin{align*}
a_0 & \neq 0, \\
a_1 & = \sqrt{3}, \\
a_n a_m & = a_{n+m} + a_{n-m} \quad \text{for all } n \geq m \geq 0.
\end{align*}
\]

How many distinct real numbers appear in the sequence?

(a) 4  
(b) 6  
(c) 7  
(d) 12  
(e) 14
28. In the diagram below,

- Both $ABCD$ and $EFGH$ are squares.
- The points $A, B, C, G, H,$ and $D$ are on a circle of radius 1.
- The points $D, E, F,$ and $C$ are collinear (they lie on the same line).

What is the area of the square $EFGH$?

(a) $\frac{1}{16}$  
(b) $\frac{2}{25}$  
(c) $\frac{2\sqrt{3} + 3}{100}$  
(d) $\frac{\sqrt{3} + 1}{25}$  
(e) $\frac{3\sqrt{2} + 4}{100}$

29. Let $E$ be a sphere, and let $S$ be a set of points in space on the outside of $E$ with the following property: For any point $P$ on $E$, there exists a point $Q$ in $S$ such that the interior of the straight line segment $PQ$ does not intersect the sphere. What is the minimum number of points in the set $S$?

Remark: It might be easier to visualize the problem with $E$ as the surface of the Earth and $S$ as a collection of satellites. What is the least number of satellites needed so that each point on Earth has unobstructed view to at least one satellite?

(a) 2  
(b) 3  
(c) 4  
(d) 6  
(e) 8

30. Exactly one of the five numbers listed below is a prime number. Which one is the prime number?

(a) 999,991  
(b) 999,973  
(c) 999,983  
(d) 1,000,001  
(e) 7,999,973