Problem 1.
Give an example of two $4 \times 4$ matrices $A$ and $B$ with real entries such that $A$ and $B$ are not similar but they do have the same minimal polynomial and the same characteristic polynomial.

Problem 2.
Suppose that $V$ is a vector space with basis $B$ and subspaces $W_0$ and $W_1$ such that $V = W_0 + W_1$.
(a) Must there be subsets of $B$ that are bases of $W_0$ and $W_1$? If so, prove it. If not, provide a counterexample.
(b) Must $V$ have a basis of the form $C \cup B_0 \cup B_1$ such that $C$ is a basis for $W_0 \cap W_1$, $B_0$ is a basis for $W_0$, and $B_1$ is a basis for $W_1$? If so, prove it. If not, provide a counterexample.

Problem 3.
$\mathbb{Z}[\sqrt{-3}]$ denotes the smallest subring of the complex numbers that contains a square root of $-3$.
(a) Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a unique factorization domain. [Hint: factor 4.]
(b) Give an example of two integral domains $D$ and $D'$ such that $D$ is a unique factorization domain and $D'$ is a homomorphic image of $D$, but $D'$ is not a unique factorization domain.

Problem 4.
Prove that $\mathbb{Q}[x]/(3x^3+2x^2+4x+6)$ is a field. Every element of this field has the form $f(x)/(3x^3+2x^2+4x+6)$ where $f(x) \in \mathbb{Q}[x]$. Find $g(x) \in \mathbb{Q}[x]$ such that $g(x)/(3x^3+2x^2+4x+6)$ is the multiplicative inverse of $x/(3x^3+2x^2+4x+6)$.

Problem 5.
Let $p$ and $q$ be prime numbers with $q < p$ such that $q$ does not divide $p^2 - 1$. Prove that each group of order $p^2q$ is Abelian.

Problem 6.
Let $R$ be a nontrivial commutative ring such that every proper ideal of $R$ is prime. Prove $R$ is a field.

Problem 7.
Let $W$ be a subspace of the vector space $V$. Let $V^*$ denote the dual space of $V$ and let $W^\circ$ denote the annihilator of $W$.
(a) Prove that $W^\circ$ is a subspace of $V^*$.
(b) Prove that $(V/W)^* \cong W^\circ$.
(c) Prove that $V^*/W^\circ \cong W^*$.

Problem 8.
Let $G$ be a finite group, let $H$ be a subgroup of $G$, and let $N$ be a normal subgroup of $G$. Suppose that $|N|$ and $[G : N]$ are relatively prime and that $|H|$ divides $|N|$. Prove that $H \subseteq N$.

Problem 9.
Let $T$ be a self-adjoint linear operator on a finite dimensional complex inner product space.
(a) Define self-adjoint both in operator terms and in matrix terms.
(b) Prove that all the coefficients of the characteristic polynomial of $T$ are real.

Problem 10.
Let $G$ and $H$ be finite Abelian groups that have the same number of elements of order $n$, for each natural number $n$. Prove that $G$ and $H$ are isomorphic.