Qualifying Exam in Linear Algebra  January 2008

1. Let $V$ be a vector space.
   a. If $\{u_1, \ldots, u_n\}$ is a spanning set for $V$ and $v \in V$ is a nonzero vector, show that there exists an $i \in \{1, \ldots, n\}$ such that $\{u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_n\}$ is also a spanning set for $V$.
   b. If $\{u_1, \ldots, u_m\}$ is a linearly independent set, and $v \in V$ is a nonzero vector, is it true that there must exist an $i \in \{1, \ldots, m\}$ such that $\{u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_m\}$ is also linearly independent? Prove or give a counterexample.

2. Let $V$ be a finite dimensional vector space over a field $F$ and $A : V \to V$ a linear transformation.
   a. Prove that there exists a positive integer $k$ such that $\ker(A^k) = \ker(A^n)$ for all $n \geq k$.
   b. Prove that there exists a positive integer $l$ such that

   $$V = \ker(A^l) \oplus \text{range}(A^l).$$

3. Let $V,W$ be finite dimensional vector spaces over a field $F$ and let $T : V \to W$ be a linear transformation, and let $T^* : W^* \to V^*$ be the dual linear transformation. Let $S \subset V$ be a subset; recall that the annihilator $S^\circ$ of $S$ is defined as

   $$S^\circ = \{ f \in V^* \mid f(s) = 0 \ \forall \ s \in S \}$$

   a. Prove that $S^\circ$ is a subspace of $V^*$.
   b. If $S \subset V$ is a subspace, prove that $\dim(S^\circ) = \dim(V) - \dim(S)$.
   c. Prove that $\ker(T)^\circ = \text{range}(T^*)$.

4. Let $A$ be a $3 \times 3$ matrix over the real numbers and assume that $f(A) = 0$, where $f(x) = x^2(x - 1)^2(x - 2)$. Give a complete list of the possible values of $\det(A)$. Justify. For each possible value on your list, give an example of a $3 \times 3$ matrix that has that determinant and satisfies the given equation.