Qualifying Examination in Linear Algebra
August 2008, Math700 and Math 706

Please use only one side of the paper and start each problem on a new page.

The field of complex numbers is denoted by \( \mathbb{C} \), the field of rational numbers by \( \mathbb{Q} \), and the ring of integers is denoted by \( \mathbb{Z} \).

1. Let \( V \) and \( W \) be vector spaces over the field \( \mathbb{F} \). Let \( v_1, \ldots, v_k \in V \) and \( w_1, \ldots, w_k \) vectors in \( W \) such that if \( a_1, \ldots, a_k \in \mathbb{F} \) then
\[
a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \quad \implies \quad a_1w_1 + a_2w_2 + \cdots + a_kw_k = 0.
\]
   (a) Show there is a linear map \( T: V \to W \) such that \( T(v_i) = w_i \) for \( i = 1, 2, \ldots, k \).
   (b) What conditions on \( v_1, v_2, \ldots, v_k \) imply that \( T \) is unique?

2. Let \( V \) be a finite dimensional complex inner product vector space. If \( W \) is a subspace of \( V \), the orthogonal complement of \( W \) is denoted by \( W^\perp \). If \( T: V \to V \) is linear, the adjoint of \( T \) is denoted by \( T^* \).
   (a) Show that a subspace \( W \) of \( V \) is \( T \)-invariant by if and only if \( W^\perp \) is \( T^* \)-invariant.
   (b) Show that if \( \langle T(x), x \rangle = 0 \), for all \( x \in V \), then \( T = 0 \).

3. Find all possible rational and Jordan canonical forms of a \( 4 \times 4 \) matrix over \( \mathbb{C} \) with characteristic polynomial \( (x - 1)^2(x + 1)^2 \) and with \( \text{rank}(A + I) = 3 \).

4. Let \( T \) be a linear operator on a vector space \( V \) and \( Z(\alpha, T) = \text{Span}(\alpha, T(\alpha), T^2(\alpha), \cdots) \) be the \( T \)-cyclic subspace of \( V \) generated by \( \alpha \). Let \( V_i = Z(\alpha_i, T) \) and \( P_i(x) \) be the characteristic polynomials of \( T|_{V_i} \). Show that if \( P_1(x) \) and \( P_2(x) \) are relatively prime, then \( Z(\alpha_1 + \alpha_2, T) = V_1 \oplus V_2 \).
Math706 – Numerical Linear Algebra

Qualifying Exam

August, 2008

Note: You must show all of your work to get a credit for a correct answer.

1. (a) For the matrix
   \[ A = \begin{bmatrix} 5.2 & 0.6 & 2.2 \\ 0.6 & 6.4 & 0.5 \\ 2.2 & 0.5 & 4.7 \end{bmatrix}, \]
   compute an upper bound for the condition number \( \kappa_2(A) \), using the estimates of
   the eigenvalues by the theorem of Gershgorin.

   (b) Find the Householder reflector that maps the vector \[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]
   to \[ \begin{bmatrix} -1 \\ 0 \\ \sqrt{2} \end{bmatrix}. \]

2. State whether the following algorithm is backward stable, stable but not backward stable,
   or unstable, and prove it or at least give a reasonably convincing argument –
   Data: none. Solution: \( e \), computed by summing \( \sum_{k=0}^{\infty} 1/k! \) from left to right using \( \otimes \)
   and \( \Theta \), stopping when a summand is reached of magnitude \( < \epsilon_{\text{machine}} \).

3. (a) Sketch the proof that every Hermitian, positive definite matrix \( A \) (i.e., \( x^*Ax > 0 \)
   for all \( x \neq 0 \)) has a unique Cholesky factorization (i.e, \( A = R^*R \) with \( r_{jj} > 0 \)).

   (b) Find the Cholesky factorization of
   \[ A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 10 & 7 \\ 2 & 7 & 17 \end{bmatrix} \]

4. Compute one step of the QR algorithm (for computing eigenvalues) with the matrix
   \[ A = \begin{bmatrix} 2 & \epsilon \\ \epsilon & 1 \end{bmatrix}. \]
   (a) Without shift.
   (b) With shift \( \mu = 1 \).

5. (a) Let \( A \in \mathbb{R}^{m \times m} \) be spd. Assume that there are exactly \( k \leq m \) distinct eigenvalues
   of \( A \). Show that the CG method for solving a linear system with the coefficient
   matrix \( A \) will terminate in at most \( k \) iterations.

   (b) Let the matrix \( A \) have the form \( A = \begin{bmatrix} I & Y \\ Y & I \end{bmatrix} \). Assume that GMRES is used to
   solve a linear system with the coefficient matrix \( A \). What is maximum number of steps that GMRES would require to converge? Explain in details.