Qualifying Examination in Algebra  
August 2008, Math 700 and Math 701

Please use only one side of the paper and start each problem on a new page.

The field of complex numbers is denoted by $\mathbb{C}$, the field of rational numbers by $\mathbb{Q}$, and the ring of integers is denoted by $\mathbb{Z}$. If $S$ is a set $|S|$ is the number of elements in $S$.

1. Let $A$ be an $4 \times 4$ matrix over $\mathbb{Q}$ with characteristic polynomial $x^4 + 15x^2 - 25x + 5$
   (a) Show that every non-zero vector in $\mathbb{Q}^4$ is cyclic under $A$.
   (b) Find the rational canonical form of $A$ over $\mathbb{C}$.

2. Let $p$ be a prime. Up to isomorphism, find all finite Abelian groups $A$ such that $|A| = p^5$ and all elements of $A$ have order at most $p^3$.

3. Give an explicit construction of a field with 9 elements.

4. Let $G$ be a finite group with $|G| = p^n$ where $p$ is a prime and $n$ is a positive integer. For $1 \leq k \leq n - 1$ let
   $s_k =$ number of subgroups of $G$ of order $p^k$,
   $n_k =$ number of normal subgroups of $G$ of order $p^k$.
   Show that $(s_k - n_k)$ is divisible by $p$.

5. Show that the quotient ring $\mathbb{Q}[x]/(x^2 - 3x + 2)$ is isomorphic to the direct product $\mathbb{Q} \times \mathbb{Q}$.

6. Let $V$ and $W$ be vector spaces over the field $F$. Let $v_1, \ldots, v_k \in V$ and $w_1, \ldots, w_k$ vectors in $W$ such that if $a_1, \ldots, a_k \in F$ then
   $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0 \implies a_1w_1 + a_2w_2 + \cdots + a_kw_k = 0$.
   (a) Show there is a linear map $T : V \rightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$.
   (b) What conditions on $v_1, v_2, \ldots, v_k$ imply that $T$ is unique?

7. Let $F(x, y)$ be the free group on the generators $x$ and $y$.
   (a) Show that $F(x, y)$ has a normal subgroup $N$ such that $F(x, y)/N$ is isomorphic to $S_3$ (the symmetric group on three elements).
   (b) Show that $F(x, y)$ has no normal subgroup $N$ such that $F(x, y)/N$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (where $\mathbb{Z}_2$ the cyclic group of order 2).

8. Let $R$ be a commutative ring with unit and $a \in R$ be an element with $a^n \neq 0$ for all positive integers. Let $I$ be an ideal of $R$ maximal with respect to the property that $a^n \not\in I$ for all $n$. Show that $I$ is a prime ideal.

9. Let $V$ be a finite dimensional real inner product vector space with inner product $\langle , \rangle$. If $W$ is a subspace of $V$, the orthogonal complement of $W$ is the denoted by $W^\perp$. If $T : V \rightarrow V$ is linear, the adjoint of $T$ is denoted by $T^*$. Show that a subspace $W$ of $V$ is invariant by $T$ if and only if $W^\perp$ is invariant by $T^*$.

10. Find all possible rational and Jordan canonical forms of a $4 \times 4$ matrix over $\mathbb{C}$ with characteristic polynomial $(x - 1)^2(x + 1)^2$ and with rank$(A + I) = 3$. 