Problem 1.
Let $G$ and $H$ be finite Abelian groups such that $G \times G \cong H \times H \times H$. Prove that there is a finite Abelian group $K$ such that $H \cong K \times K$.

Problem 2.
Let $V$ be a vector space with subspaces $U$ and $W$, neither one contained in the other. Show that there are disjoint, non-empty sets $A \subseteq U$ and $B \subseteq W$ such that $A \cup B$ is a basis for $U + W$. Show by example that $A$ can span a proper subspace of $U$ and simultaneously $B$ span a proper subspace of $W$, while $A \cup B$ is still a basis for $U + W$.

Problem 3.
(a) Prove that $2x^3 + 6x^2 + 9x + 12$ is irreducible in $\mathbb{Z}[x]$.
(b) Prove that $x^3y + x^2y^2 + x + y^3$ is irreducible in $\mathbb{Z}[x, y]$.

Problem 4.
Prove that a group with a proper subgroup of finite index has a proper normal subgroup of finite index.

Problem 5.
Make a list, as long as possible, of square matrices over the reals such that
(1) Each matrix on the list has characteristic polynomial $(x - 2)^3(x - 3)^3$,
(2) Each matrix on the list has minimal polynomial $(x - 2)^2(x - 3)^2$, and
(3) No two distinct matrices on the list are similar.
Demonstrate that your list has all the desired attributes.

Problem 6.
Prove that a commutative ring $R$ cannot have three distinct proper nontrivial ideals $I, J,$ and $K$ such that $I \subseteq J, I + K = R,$ and $J \cap K = \{0\}$.

Problem 7.
Let $F$ be a field and let $A$ and $B$ be nonsingular (invertible) $3 \times 3$ matrices over $F$. Suppose that $B^{-1}AB = 2A$. Prove the statements below. Even if you cannot establish one part, you may use it in subsequent parts.
(a) The characteristic of $F$ is 7.
(b) $A$ has trace 0.
(c) $A^3 = aI$ for some scalar $a \in F$.

Problem 8.
Prove that every group of order 18 which has a normal subgroup of order 2 is an Abelian group.

Problem 9.
Let $R$ be a commutative ring. Prove that $R$ has no infinite ascending chain $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ of ideals if and only if every ideal of $R$ is finitely generated.

Problem 10.
Let $T$ be a linear operator on the inner product space $V$ such that $T$ has an adjoint $T^*$. Let $W$ be a $T$-invariant subspace of $V$. (So $T_W$, the restriction of $T$ to $W$, is a linear operator on $W$). Assume $T_W$ also has an adjoint $(T_W)^*$. Recall $W^\perp = \{ z \in V \mid \langle y, z \rangle = 0 \text{ for all } y \in W \}$.
(a) Prove $W^\perp$ is $T^*$-invariant.
(b) Prove that if $W$ is both $T$ and $T^*$-invariant, then $(T_W)^* = (T^*)_W$.
(c) Assume further that $V$ is a finite dimensional complex vector space and that $T$ is a normal operator. Prove $T_W$ is a normal operator on $W$. 