Qualifying Exam in Linear Algebra  August 2007

Each problem is worth 10 points.

1. a. Let $V$ be a vector space. If $S \subseteq V$ is a spanning set for $V$, prove that there exists $S' \subseteq S$ which is a basis for $V$. (This is a well-known theoretical fact, but I am requiring a proof of it.)

b. TRUE/FALSE: If $V$ is a vector space and $W \subseteq V$ is a subspace, then for every basis $B$ of $V$, there exists a subset $B' \subseteq B$ such that $B'$ is a basis for $W$. If true, prove it. If false, give a counterexample.

2. Let $V$ be a vector space.

a. If $T : V \to V$ is a linear transformation such that $T^2 = T$, prove that $V = \text{Ker}(T) \oplus \text{Im}(T)$.

b. If $V = \mathbb{R}^3$, give an example of a linear transformation $T$ as in part (a) which is not 0 and is not the identity.

c. If $U, W \subseteq V$ are subspaces such that $V = U \oplus W$, prove that there exists a linear transformation $T : V \to V$ such that $T^2 = T$, $U = \text{Ker}(T)$, and $W = \text{Im}(T)$.

3. Let $V, W$ be arbitrary vector spaces, and let $T : V \to W$ be a linear transformation. Define $T^* : W^* \to V^*$ by $T^*(f) = f \circ T$ for all $f \in W^*$.

a. Prove or give a counterexample:

"$T^*$ is one-to-one if and only if $T$ is onto."

b. Prove or give a counterexample:

"$T^*$ is onto if and only if $T$ is one-to-one."

4. Let $A, B$ be $n \times n$ matrices over $\mathbb{C}$ with $AB = BA$.

a. Prove that $A$ and $B$ have a common eigenvector.

b. Must $A, B$ have a common eigenvalue? Prove or give a counterexample.

5. Let $T$ be a linear operator on a complex inner product space $V$, and assume that the adjoint operator $T^{adj}$ exists

(so that $<T(x), y> = <x, T^{adj}(y)>$ for all $x, y \in V$).

a. If $<T(x), x> = 0$ for all $x \in V$, prove that $T = 0$.

b. Prove that $T = T^{adj}$ if and only if $<T(x), x>$ is real for all $x \in V$.

6. Let $P_2$ be the vector space over $\mathbb{C}$ of all polynomials of degree $\leq 2$. Find the Jordan canonical form of $T$, where $T : P_2 \to P_2$ is defined by

$$(T(p))(x) = (x + 1)^2 p \left( \frac{x - 1}{x + 1} \right).$$
Note! You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet; start each problem on a new sheet of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc. If some problem is incorrect, then give a counterexample. “Describe” means give the best possible description.

The exam is worth 60 points. There are five problems.

1. (10 points) Let $K$ be a field and $G$ be a finite subgroup of the multiplicative group $K \setminus \{0\}$. Describe the structure of $G$. Prove your answer.

2. Let $I$ be an ideal in a commutative ring $R$ and let $\mathcal{S}$ be the set of ideals of $R$ defined by the property that $J \in \mathcal{S}$ if and only if there exists an element $a$ of $R$ such that $a \notin I$ and $J = \{r \in R \mid ra \in I\}$.
   - (a) (10 points) Prove that every maximal element of $\mathcal{S}$ is a prime ideal in $R$.
   - (b) (10 points) Let $R$ be the ring of integers and $I$ be the ideal $(36)$ in $R$. Identify all of the elements of $\mathcal{S}$. Prove your answer.

3. (10 points) Let $H$ be a normal subgroup of a finite group $G$, let $K$ be a Sylow subgroup of $H$, and $L$ be the normalizer of $K$ in $G$; that is, $L = \{g \in G \mid gKg^{-1} \subseteq K\}$. Does $LH$ have to equal $G$? Prove or give a counter example.

4. (10 points) Let $a_1, \ldots, a_n$ be distinct elements of the field $K$ and for each $i$ let $f_i$ be the polynomial $f_i = \prod_{j \neq i} (x - a_j)$. Describe the ideal $(f_1, \ldots, f_n)$ of the polynomial ring $K[x]$. Prove your answer.

5. (10 points) Let $V$ be the subgroup $\{(12)(34), (13)(24), (14)(23), (1)\}$ of the group $S_4$. Exhibit an isomorphism between $S_4/V$ and some well-known uncomplicated group. Give the best possible argument; but give all details.