PROBLEM 1.  
Characterize up to similarity those $3 \times 3$ matrices $A$ over the real numbers such that $f(A) = 0$ where $f(x) = x(x - 1)^2(x - 2)^3$.

PROBLEM 2.  
Characterize up to isomorphism all Abelian groups of order 600.

PROBLEM 3.  
(a) Prove that $3x^3 + 6x^2 + 12x + 14$ is irreducible in $\mathbb{Z}[x]$.  
(b) Prove that $x^2y^3 + x^2y^2 + x^2y - 2xy^2 + x + y^3 + y^2 - y - 1$ is irreducible in $\mathbb{Z}[x,y]$.

PROBLEM 4.  
Let $A$ be a non-zero $n \times n$ matrix over a field $F$. Prove each of the following.  
(a) If $A^{-1}$ exists, then $A^{-1}$ can be written as a polynomial in $A$.  
(b) If $A^{-1}$ does not exist, then there is an integer $k > 0$ and a non-zero $n \times n$ matrix $B$ such that $A^kB = BA^k = 0$.

PROBLEM 5.  
Let $G$ be a group. Prove that $G$ is finite if and only if $G$ has only finitely many subgroups.

PROBLEM 6.  
Prove that the ring $\mathbb{Z}_{10}$ and the ring $\mathbb{Z}_{10} \times \mathbb{Z}_{21}$ are isomorphic.

PROBLEM 7.  
Let $V$ be vector space and let $\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ be a chain of $n + 1$ distinct subspaces of $V$ so that for each $j < n$ there is no subspace which lies properly between $V_j$ and $V_{j+1}$. That is, this is a maximal chain of subspaces of $V$. Let $W$ be a subspace of $V$ and put $W_j = V_j \cap W$ for all $j \leq n$. Evidently, $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_n = W$ is a chain of subspaces of $W$.  
(a) Determine the dimension of each $V_j$, and produce a basis $\{v_1, v_2, \ldots\}$ for $V$ such that $v_i \in V_i$.  
(b) Give an explicit example to show that the subspaces $W_0, W_1, \ldots, W_n$ need not be distinct.  
(c) Prove that $W_0 \subset \cdots \subset W_n = W$ is a maximal chain of subspaces of $W$.

PROBLEM 8.  
Let $p, q \in \mathbb{Z}$ be primes with $q \leq p$ and $p \neq 1 \mod q$. Prove that any group of order $pq$ is Abelian.

PROBLEM 9.  
Let $T$ be a linear operator on a complex vector space $V$, and assume that $T^\text{adj}$ exists.  
(a) Prove that if $T$ is self-adjoint (Hermitian), then $\langle Tx, x \rangle$ is real for all $x \in V$.  
(b) Prove the converse of (a). You may use the fact that $\langle Tx, x \rangle = 0$ for all $x \in V$ implies that $T = 0$, the zero operator.  
(c) Prove that if $V$ is finite dimensional, $T$ is self-adjoint, and $A = [T]_\beta$ is a matrix representation of $T$ with respect to some orthonormal basis $\beta$ of $V$, then there is a a real diagonal matrix $D$ and a unitary matrix $U$ so that $A = U^\text{adj}DU$. Do not just say that this is a theorem in the book, but explain how $D$ and $U$ arise from $T$ or $A$ (quoting theorems as you need them).

PROBLEM 10.  
Prove that in a principal ideal domain every nontrivial prime ideal is maximal. Give an example of an integral domain with a nontrivial prime ideal that is not maximal.