Qualifying Exam for Linear and Numerical Linear Algebra

August 2005

Do all nine problems. You can use a calculator for problem 5 and problem 6.

1. Prove that if $U$ and $W$ are subspaces of the vector space $V$ such that $U + W = V$, then there is a subspace $W'$ of $W$ such that $U \oplus W' = V$.

2. Let $V$ be a vector space and $T : V \to V$ be a linear transformation that is neither an isomorphism nor the zero transformation.
   
   (a) Prove that if $V$ is finite dimensional then there is a linear transformation $S : V \to V$ such that $S \circ T$ is the zero transformation but $T \circ S$ is not the zero transformation.
   
   (b) Does the statement asserted in (a) remain true when the hypothesis that $V$ is finite dimensional is dropped? Prove it or give a counterexample.

3. Prove that if $A$ is an $n \times n$ nilpotent matrix, then $A^n = 0$.

4. Let $P_3$ be the vector space over the field $\mathbb{C}$ of complex numbers of all polynomials over $\mathbb{C}$ with degree no more than 3. Let $T$ be the map on $P_3$ defined by

   $$(T(p))(z) = (z + 1)^3 p \left( \frac{z}{z+1} \right)$$

   for all $p \in P_3$.

   Prove that $T$ is a linear transformation from $P_3$ into $P_3$ and find its Jordan canonical form.

5. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & -4 \\ 2 & -4 & 14 \end{bmatrix}$ and $b = \begin{bmatrix} -5 \\ 23 \\ -10 \end{bmatrix}$

   (a) Show that $A$ is symmetric positive definite and find the Cholesky factorization of $A$.
   
   (b) Use the Cholesky factorization from (a) to solve $Ax = b$. 
6. Let \( A = \begin{bmatrix} -7 & 75 \\ 4 & 24 \\ -4 & -87 \end{bmatrix} \) and \( b = \begin{bmatrix} 75 \\ 141 \\ -51 \end{bmatrix} \).

(a) Find a reduced QR factorization of \( A \) using Householder reflectors.

(b) Use the QR factorization from (a) to find the least squares solution of \( Ax = b \).

7. Let \( n \geq m \) and \( A \in \mathbb{R}^{n \times m} \). Show that there exists a bidiagonal matrix \( B \in \mathbb{R}^{m \times m} \) such that \( B \) and \( A \) have identical singular values.

8. Show that back substitution is backward stable for \( 2 \times 2 \) upper triangular systems.

9. Let \( T \in \mathbb{R}^{n \times n} \) be a symmetric tridiagonal matrix such that \(|T(n - 1, n)| \leq \epsilon\), where \( T(n - 1, 1) \) is the \(((n - 1), n)\) component of \( T \). Show that

(a) the component \( T(n, n) \) of \( T \) is within a distance of \( \epsilon \) from an eigenvalue of \( T \),

(b) any eigenvalue of the sub-matrix \( T(1 : n-1, 1 : n-1) \) is within a distance of \( \epsilon \) from an eigenvalue of \( T \).