Instructions: Write your name legibly on each sheet of paper. Write only on one side of each sheet of paper. Try to answer all questions. Questions 1-8 are each worth 10 points and question 9 is worth 20 points.

Terminology: Measurability and integrability on \( \mathbb{R} \) or an interval will always refer to the Lebesgue measure, except if otherwise specified. Lebesgue measure will be denoted by \( m \), \( dx \) or \( dy \) depending on the context.

1. Let \( f_n \) be absolutely continuous on \([0,1]\) and let \( f_n(0) = 0 \). Assume that
\[
\int_0^1 \left| f'_n(x) - f'_m(x) \right| \, dx \to 0
\]
as \( m, n \to \infty \). Prove that \( f_n \) converges uniformly to a function \( f \) on \([0,1]\) and that \( f \) is absolutely continuous on \([0,1]\).

2. Let \( f \) be a non-negative measurable function on \([0,1]\). Prove that
\[
\int_0^1 f(x) \, dx
\]
exists in \( \mathbb{R} \) if and only if \( m\{x : f(x) > 1\} = 0 \).

3. Let \( A \) be a Lebesgue measurable subset of \( \mathbb{R}^2 \) with \( m \times m(A) > 0 \) and let \( \{l_i : i \in I\} \) be a collection of lines in \( \mathbb{R}^2 \) such that \( A \subseteq \bigcup_{i \in I} l_i \). Prove that \( I \) is uncountable.

4. Let \( f \in L^2([-1,1]) \). Prove that
\[
\int_{-1}^1 \frac{|f(x)|}{\sqrt{|x|}} \, dx < \infty.
\]

5. Let \( f_n \) be measurable functions on \([0,1]\) such that \( f_n(x) \to f(x) \) a.e. Assume that \( f(x) \neq 0 \) a.e. Prove that for all \( \epsilon > 0 \) there exists \( c > 0 \), a measurable set \( E \subseteq [0,1] \) and \( N \in \mathbb{N} \) such that \( |f_n(x)| \geq c \) on \( E \) for all \( n \geq N \) and such that \( m([0,1] \setminus E) < \epsilon \).

6. Let \( G \subseteq \mathbb{C} \) be a region and let \( f : G \to \mathbb{C} \) be a holomorphic function such that \( |f(z)| = C \) for all \( z \in G \). Prove that \( f \) is constant on \( G \).

7. Let \( f \) be a holomorphic function defined in a neighborhood of the origin and let \( f'(0) \neq 0 \).

(a) Show that there exists \( r > 0 \) such that the unique solution of the equation \( f(z) = f(0) \) in the disc \( |z| < r \) is \( z = 0 \).
(b) Prove that if $r > 0$ is sufficienty small, then

$$\frac{1}{f''(0)} = \frac{1}{2\pi i} \int_{|z|=r} \frac{1}{f(z) - f(0)} \, dz,$$

where the circle $|z| = r$ is traversed counterclockwise.

8. Let $f : \{z : |z| < 1\} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq 1$ for all $|z| < 1$, $f(0) = 0$, and $|f(\frac{1}{2})| = \frac{1}{4}$. Prove that there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for all $|z| < 1$.

9. True or False. Prove, or give a counterexample.

   a. If $m^*(A) = 0$, then $m^*(B \setminus A) = m^*(B)$ for all subsets $B$ of $\mathbb{R}$.

   b. Let $Q = \{r_n : n = 1, 2, \ldots\}$. Then

   $$\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \left( r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2} \right) \neq \emptyset.$$

   c. The function $f(z) = e^z$ is analytic everywhere except at 0.

   d. Let $f$ be an entire function such that $\lim_{z \to \infty} |f(z)| = \infty$. Then $f$ is a polynomial.

   e. Let $f_n$ be integrable functions on $\mathbb{R}$ such that $\lim_{n \to \infty} \int f_n \, dx = 0$. Then there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and an integrable function $g$ such that $|f_{n_k}| \leq g$ a.e. for all $k$. 

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