The field of complex numbers is denoted by $\mathbb{C}$ and the ring of integers is denoted by $\mathbb{Z}$.

1. Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most three over the complex numbers $\mathbb{C}$. Define a linear map $T: \mathcal{P}_3 \to \mathcal{P}_3$ by
   \[ Tp(x) = p(x + 2). \]

   Find
   \begin{enumerate}
   \item The trace of $T$,
   \item The determinant of $T$, and
   \item The Jordan canonical form of $T$.
   \end{enumerate}

2. Classify up to isomorphism all groups of order 45.

3. Let $G$ be a finite group and $Z(G)$ the center of $G$. Then show that $G/Z(G)$ is never a cyclic group with more than one element.

4. Show that the quotient ring $\mathbb{C}[x]/(x^2+1)$ is isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$. Here $(x^2+1)$ is the ideal generated by $x^2 + 1$.

5. Let $V$ be a finite dimensional vector space over the field $\mathbb{F}$ and let $V^*$ be the dual space of $V$. Let $S$ be a non-empty subset of $V$ and let
   \[ W = \{ w \in V : \text{for all } f \in V^* \text{ if } f(v) = 0 \text{ for all } v \in S, \text{ then } f(w) = 0 \}. \]

   Show that $W$ is span of $S$.

6. Let $G$ and $H$ be finite Abelian groups with $H \times H \cong G \times G$. Show that $H \cong G$.

7. Let $D$ be a commutative ring with identity such that $D[x]$ is a principal ideal domain. Show that $D$ is a field.

8. Let $A$ be an $n \times n$ matrix with $A^3 = 4A$. Show that
   \[ -2 \text{ rank}(A) \leq \text{ trace}(A) \leq 2 \text{ rank}(A). \]

9. \begin{enumerate}
   \item Show that $x^3y + x^3 + x^2y^2 - x^2y + xy^2 + y$ is irreducible in $\mathbb{Z}[x,y]$.
   \item Show that $5x^4 - 2x^2 + x + 15$ is irreducible in $\mathbb{Z}[x]$.
   \end{enumerate}

10. Let $P$ be a Sylow subgroup of the finite group $G$ and let $H$ be a subgroup of $G$ with $N_G(P) \leq H$. Prove $H = N_G(H)$. 

Qualifying Examination in Algebra  
August 2003

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Find

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(b) The determinant of \( T \), and
(c) The Jordan canonical form of \( T \).

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3. Let \( G \) be a finite group and \( Z(G) \) the center of \( G \). Then show that \( G/Z(G) \) is never a cyclic group with more than one element.

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\[ -2 \text{ rank}(A) \leq \text{ trace}(A) \leq 2 \text{ rank}(A). \]

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