QUALIFYING EXAM COVERING MATH 700 AND MATH 706

Instructions: Prove all your assertions; quote big theorems by name.

Problem #1. Let $V = \mathbb{R}^{n \times n}$ be the vector space of all $n \times n$ matrices with entries in $\mathbb{R}$. Consider the linear operator $S: V \to V$ given by $S(A) = (A + A^T)/2$, where $A^T$ is the transpose of $A$. Compute the rank and nullity of $S$.

Problem #2. Find a set $S$ of $n \times n$ matrices over $\mathbb{C}$ that has the following properties:

1. the characteristic polynomial of each $A \in S$ is $(x + 1)^4(x - i)^2$;
2. the minimal polynomial of each $A \in S$ is $(x + 1)^2(x - i)$;
3. each $A \in S$ has the same value of $n$, and every $n \times n$ matrix $A'$ over $\mathbb{C}$ satisfying (1) and (2) is similar to one and only one matrix in $S$.

Problem #3.

1. Show that if $A$ and $B$ are in $\mathbb{C}^{3 \times 3}$ then $AB - BA \neq I_{3 \times 3}$ (the identity matrix).
2. Show that if both $A$ and $B$ are diagonalizable and $AB = BA$ then there is an invertible matrix $P$ such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal. Hint: Find $X$ such that $X^{-1}AX$ is diagonal and $X^{-1}BX$ is block diagonal, and show that each of the blocks of $X^{-1}BX$ is diagonalizable (consider their minimal polynomials).

Problem #4. State the Cayley-Hamilton Theorem.

Problem #5. Show that every nilpotent matrix with entries in an arbitrary field is similar to a strictly upper triangular matrix.

Problem #6. State and prove a theorem on the existence of a singular value decomposition of a matrix.

Problem #7. Define the Frobenius norm $\|A\|_F$ of a matrix and show that $\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_{2,2}$. ($\|A\|_{2,2}$ denotes the 2-norm of $A$.)

Problem #8. If $A$ is a nonsingular real $n \times n$ matrix show that there exists a permutation matrix $P$ such that $PA$ has an $LU$ factorization, where all the entries of $L$ are in the interval $[-1, 1]$.

Problem #9. Find a $3 \times 3$ Householder reflection $H$ such that $Hx = (x_1, \sqrt{x_2^2 + x_3^2}, 0)^T$ when $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ and $x_3 \neq 0$. If $A$ is an $m \times n$ real matrix with $m \geq n$, and rank($A$) = $n$, show that there exists an $m \times m$ orthogonal matrix $Q$ and an $m \times n$ upper triangular matrix $R$ such that $A = QR$.

Problem #10. Suppose $A$ is an $n \times n$ complex matrix. Show that there exists a unitary matrix $Q$ and an upper triangular matrix $T$ such that $AQ = QT$.

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