Qualifying Exam in Linear Algebra and Numerical Linear Algebra

August, 2002

Linear Algebra

Do all five problems.

1. Let $V$ be the vector space of all $n \times n$ matrices over a field of characteristic $p \neq 2$. A linear operator $T$ on $V$ is defined by $T(A) = A + A^t$.
   (a) Find the rank and the nullity of $T$.
   (b) Show that $V = R(T) \oplus N(T)$.

2. Let $\mathbb{F}$ be a field with $q$ elements. Find the number of invertible $n \times n$ matrices over $\mathbb{F}$.

3. Show that, over $\mathbb{C}$, a linear operator $T$ is nilpotent if and only if the trace of $T^k$ is equal to 0 for all $k$.

4. Classify up to similarity all $n \times n$ matrices $A$ such that $A^2 = A$.

5. Let $A$ be an $n \times n$ matrix over $\mathbb{Q}$ with the characteristic polynomial of $f(x) = x^n + 14x^{n-1} - 49x^2 + 21$ ($n \geq 4$). Show that
   (a) Every non-zero vector in $\mathbb{Q}^n$ is cyclic under $A$.
   (b) $A$ is diagonalizable over $\mathbb{C}$.

Numerical Linear Algebra

Do three of the following four problems.

1. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.
   (a) Find a reduced QR factorization of $A$.
   (b) Find the Cholesky factorization of $A^T A$.
   (c) Find the least squares solution of the overdetermined system $Ax = b$ using (i) the QR factorization of $A$ and (ii) the Cholesky factorization of $A^T A$ (applied to the normal equation).
   (d) For a general full rank matrix $A \in \mathbb{R}^{m \times n}$ ($m \gg n$) and an arbitrary $b \in \mathbb{R}^m$, which approach is more efficient for finding the least squares solution of $Ax = b$?