The field of real numbers will be denoted by \( \mathbb{R} \), the field of complex numbers by \( \mathbb{C} \) and the field of rational numbers by \( \mathbb{Q} \). The ring of integers is denoted by \( \mathbb{Z} \).

**Note!** You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet; start each problem on a new sheet of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc. If some problem is incorrect, then give a counterexample.

1. Prove that if \( k \) is a field and \( f(x) \) and \( g(x) \) are relatively prime elements of \( k[x] \), then the rings
\[
\frac{k[x]}{(f(x)g(x))} \quad \text{and} \quad \frac{k[x]}{(f(x))} \times \frac{k[x]}{(g(x))}
\]
are isomorphic.

2. Let \( V \) be the vector space of all \( n \times n \) matrices over the field \( \mathbb{Q} \). A linear operator \( T \) on \( V \) is defined by \( T(A) = A + A^t \).
   (a) Find the rank and the nullity of \( T \).
   (b) Show that \( V = R(T) \oplus N(T) \).

3. Let \( f \) and \( g \) be polynomials in an indeterminate over a ring \( R \). Suppose that the ideal generated by the coefficients of \( f \) is \( R \) and that the ideal generated by the coefficients of \( g \) is \( R \). Prove that the coefficients of \( fg \) also generate \( R \).

4. Classify up to similarity all \( n \times n \) matrices \( A \) such that \( A^2 = A \).

5. Let \( F \) be a field with \( q \) elements. Find the number of invertible \( n \times n \) matrices over \( F \).

6. Let \( R \) be a commutative ring and suppose that \( I \) is maximal among all the ideals of \( R \) which are not principal. Prove that \( I \) is a prime ideal.

7. Let \( K \) be a normal subgroup of a finite group \( G \). If \( P \) is a sylow \( p \)-subgroup of \( K \) for some \( p \), prove that \( G = KNG(P) \). (Recall that the normalizer of \( P \) in \( G \) is equal to \( N_G(P) = \{ g \in G \mid gPg^{-1} \subseteq P \} \). Recall, also that the product of two subgroups \( A \) and \( B \) of \( G \) is the set of all \( ab \) with \( a \in A \) and \( b \in B \).)

8. Let \( A \) be an \( n \times n \) matrix over \( \mathbb{Q} \) with the characteristic polynomial of \( f(x) = x^n + 14x^{n-1} - 49x^2 + 21 \) with \( 4 \leq n \).
   (a) Prove that every non-zero vector in \( \mathbb{Q}^n \) is cyclic under \( A \).
   (b) Prove that \( A \) is diagonalizable over \( \mathbb{C} \).

9. Show that, over \( \mathbb{C} \), a linear operator \( T \) is nilpotent if and only if the trace of \( T^k \) is equal to 0 for all \( k \).

10. Prove that \( S_5 \) has no subgroup of order 30.