The field of real numbers will be denoted by \( \mathbb{R} \), the field of complex numbers by \( \mathbb{C} \) and the field of rational numbers by \( \mathbb{Q} \). The ring of integers is denoted by \( \mathbb{Z} \).

**Note!** You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet; start each problem on a new sheet of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc. If some problem is incorrect, then give a counterexample.

1. Classify the groups of order 33.

2. Let \( I \) be the ideal of \( \mathbb{C}[x, y, z] \) which consists of all polynomials \( g(x, y, z) \) such that \( g(1, 2, 3) = 0 \). Prove that \( I \) is a maximal ideal.

3. Let \( \mathcal{P}_2 \) be the vector space of real polynomials of degree \( \leq 2 \). Let \( \Lambda_1, \Lambda_2, \Lambda_3 : \mathcal{P}_2 \rightarrow \mathbb{R} \) be the linear functionals given by

\[
\Lambda_1(p) = p(1), \quad \Lambda_2(p) = p(-1), \quad \Lambda_3(p) = p(2).
\]

Show that \( \Lambda_1, \Lambda_2, \Lambda_3 \) is a basis of the dual space to \( \mathcal{P}_2 \) and find the basis of \( \mathcal{P}_2 \) dual to this basis.

4. Let \( \mathcal{P}_3 \) be the vector space over the complex numbers \( \mathbb{R} \) of all polynomials of degree \( \leq 3 \). Define a linear map \( T : \mathcal{P}_3 \rightarrow \mathcal{P}_3 \) by

\[
(Tp)(x) = (x + 1)^3 p \left( \frac{x}{x + 1} \right).
\]

Find the Jordan canonical form of \( T \).

5. Prove that the identity map is the only ring homomorphism from \( \mathbb{R} \) to \( \mathbb{R} \).

6. Let \( \ell \) be the line in 3-space which contains the origin and the point \((1,0,3)\), and let \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be rotation by 30 degrees about the axis described by the line \( \ell \). Find a matrix \( M \) so that

\[
f(x, y, z) = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

for all points \((x, y, z) \in \mathbb{R}^3\).

7. Suppose \( R \) is a commutative ring with 1 and an ideal \( I \) can be expressed as a product of distinct maximal ideals in two ways: \( P_1 \cdots P_n \) and \( Q_1 \cdots Q_s \). Prove that these factorizations are identical except for the order of the factors.

8. Suppose \( G \) is a finite group, \( P \) is a Sylow \( p \)-subgroup, and there is a normal subgroup \( H \trianglel G \) that contains \( P \). Show that if \( P \) is a normal subgroup in \( H \), then \( P \) is a normal subgroup in \( G \). Give an example to show that the conclusion can fail if \( P \) is not a Sylow \( p \)-subgroup.

9. Let \( G \) be a finite group of order \( 2q \) where \( q \) is an odd number. Show that \( G \) has a subgroup \( H \) of order \( q \).