Admission to Candidacy Examination in Algebra
January 2000

Note! You must show sufficient work to support your answer. Write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet and be sure to number your pages.

1. Let $A$ be a square matrix of rank 1. Prove that $A$ is diagonalizable.

2. Let $G$ be a finite group and $D$ be a division ring. Let $f : G \rightarrow D^*$ be an non-trivial group homomorphism, where $D^*$ is the multiplicative group $D \setminus \{0\}$. Prove that $\sum_{g \in G} f(g) = 0$.

3. Let $a$, $b$, and $c$ be integers. The natural map

$$
\mathbb{Z} \rightarrow \frac{\mathbb{Z}}{(a)} \oplus \frac{\mathbb{Z}}{(b)} \oplus \frac{\mathbb{Z}}{(c)}
$$

is a surjection if and only if _______. FILL in the blank with some conditions on the integers $a$, $b$, and $c$ and PROVE the resulting statement.

4. Let $G$ be a group of order $p^2q$, where $p$ and $q$ are prime integers. Prove that $G$ is not simple. (In other words, prove that $G$ has a nontrivial normal subgroup.)

5. Let $G$ be a group of order $p^n$ for some prime $p$ and let $H$ be a normal subgroup of $G$, with $H \neq \{1\}$. Prove that $Z(G) \cap H \neq \{1\}$, where $Z(G)$ is the center of $G$.

6. Identify six distinct proper prime ideals $P_0 \subseteq P_1 \subseteq P_2$, and $Q_0 \subseteq Q_1 \subseteq Q_2$ in the ring $\mathbb{Z}[x, y]/(x^2 - 4y^2)$.

7. Let $A$ be an $n \times n$ matrix over $\mathbb{Q}$ such that its characteristic polynomial is irreducible in $\mathbb{Q}[x]$. Prove that:

(a) Every non-zero vector in $\mathbb{Q}^n$ is a cyclic generator for $\mathbb{Q}^n$ under $A$.
(b) $A$ is diagonalizable over $\mathbb{C}$.

8. Let $N_1$ and $N_2$ be two $n \times n$ nilpotent matrices with the same minimal polynomial and the same nullity. Find the maximal $n$ such that $N_1$ and $N_2$ must be similar.

9. Suppose that $N$ is a normal subgroup of the finite group $G$ and that $H$ is a subgroup of $G$ with $G = NH$ and $N \cap H = \{1\}$. PROVE that THERE EXISTS a homomorphism $\theta : H \rightarrow \text{Aut} N$ such that $G \cong N \rtimes_\theta H$. Recall that multiplication in $N \rtimes_\theta H$ is given by

$$(n, h)(n', h') = (n \cdot \theta(h)|_{n'}, hh').$$

10. Let $G$ be a group and $f : G \rightarrow G$ be a function such that $f(a)f(b)f(c) = f(A)f(B)f(C)$ whenever $abc = ABC = 1$. Prove that there exists $g \in G$ such that $h(x) = gf(x)$ is a homomorphism.