Questions 1 - 8 are worth ten points each and question nine is worth 20 points.

1) Suppose that $E, F \subseteq \mathbb{R}$ are compact. Prove that the set $E + F := \{x + y : x \in E, y \in F\}$ is compact.

2) Let $f_n \in L_2[a, b]$ be such that $\sum_{n=1}^{\infty} (\int_a^b |f_n|^2) = \infty$. Show that
   (a) $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for almost all $x \in [a, b]$.
   (b) If $f(x) = \sum_{n=1}^{\infty} f_n(x)$, then $f \in L_2[a, b]$.
   (c) $(\int_a^b |f_n - f|^2)^{1/2} \to 0$.

3) Let $f_n$ be measurable on $\mathbb{R}$, $f_n \geq 0$, and $f \in L_1(\mathbb{R})$. Prove that if $f_n \to f$ a.e. on $\mathbb{R}$, then
   $\int_{\mathbb{R}} (f - f_n)^+ \to 0$, where $x^+ := \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$

4) State and prove Vitali's covering lemma.

5) Let $(g_n)$ be a sequence of measurable functions on $\mathbb{R}$ such that there exists $M > 0$ with $|g_n| \leq M$ for $n = 1, 2, \ldots$. Suppose $\int_E g_n \to 0$ for every set $E \subseteq \mathbb{R}$ such that $m(E) < \infty$. Prove that for every $f \in L_1(\mathbb{R})$
   $\int_{\mathbb{R}} fg_n \to 0$.

6) Suppose $f$ is measurable on $[a, b] \times [c, d]$, $f(x, y) \geq 0$ on $[a, b] \times [c, d]$, and
   $\int \int_{[a, b] \times [c, d]} f(x, y) \, dx \, dy > b - a$.
   Show that there exists $x \in [a, b]$ such that
   $\int_c^d f(x, y) \, dy > 1$.

7) Show that if $f \in L_p[0, 1], 1 < p < \infty$, then $f \in L_1[0, 1]$ and $\|f\|_{L_1} \leq \|f\|_{L_p}$.
   Is the result true if $[0, 1]$ is replaced by $\mathbb{R}$? Prove or give a counterexample.

8) (a) State and prove Liouville's theorem.
    (b) Suppose $f$ is an entire function and $|f(z)| \leq Me^{Rez}$ for every $z$ from the complex plane, where $M$ is a constant. Prove that there exists a constant $C$ such that $f(z) = Ce^{z}$. 
9. True or False? Prove or give a counterexample.
   (a) Every uncountable set of real numbers has a non-measurable subset.
   (b) For every $\varepsilon > 0$ there exists an open dense subset $O$ of $[0, 1]$ such that $m(O) < \varepsilon$.
   (c) If $f_n$ are measurable on $[0, 1]$, $f_n \geq 0$, $\int_0^1 f_n = 1$, and $f_n \to f$ a.e., then $\int_0^1 f = 1$.
   (d) If $f_n$ are measurable on $\mathbb{R}$, $0 \leq f_1 \leq f_2 \leq \ldots$, $f_n \to f$ a.e., and $\int_{\mathbb{R}} f_n \to 1$, then $f \in L_1(\mathbb{R})$.
   (e) Suppose $f$ and $g$ are continuous on $[0, 1]$, $f'$ and $g'$ exist a.e. on $[0, 1]$, $f' = g'$ a.e., and $f(0) = g(0)$. Then $f = g$ on $[0, 1]$. 