In the problems below, when $T$ is a linear operator on a vector space $V$, the null space of $T$ is denoted by $\text{null}T$, while $\text{range}T$ denotes $\{Tv : v \in V\}$ which is called the range space of $T$ or the image of $T$ (by some authors). Rings are defined to have a unit. We use $\mathbb{R}$ to denote the field of real numbers and $\mathbb{C}$ to denote the field of complex numbers.

Please write your answers as legibly as you can on the blank sheets of paper provided. Use only one side of each sheet and be sure to number your pages.

**PROBLEM 1.**
Let $V$ be a finite dimensional vector space and let $T : V \to V$ be a linear transformation which is not zero and is not an isomorphism. Prove that there exists a linear transformation $S : V \to V$ such that $ST = 0$ but $TS \neq 0$.

**PROBLEM 2.**
Let $T$ be a linear operator on the finite dimensional vector space $V$. Prove that if $T^2 = T$, then $V = \text{null}T \oplus \text{range}T$.

**PROBLEM 3.**
True or False. (If true, then PROVE it. If false, then give a COUNTEREXAMPLE.) If $N$ is a normal subgroup of the group $G_1 \times G_2$, and $N_1$ is the projection of $N$ into $G_1$ and $N_2$ is the projection of $N$ into $G_2$, then

$$\frac{G_1 \times G_2}{N} \cong \frac{G_1}{N_1 \cap G_1} \times \frac{G_2}{N_2 \cap G_2}.$$

**PROBLEM 4.**
Prove that an infinite integral domain with a finite number of units has an infinite number of maximal ideals.

**PROBLEM 5.**
Let $S$ and $T$ be $5 \times 5$ nilpotent matrices with $\text{rank}S = \text{rank}T$ and $\text{rank}S^2 = \text{rank}T^2$. Are $S$ and $T$ necessarily similar? Prove or give a counterexample.

**PROBLEM 6.**
Let $G$ be a finitely generated group which has exactly one maximal proper subgroup $A$. Prove that $G$ is generated by any element not in $A$. Prove that $G$ is cyclic of prime power order.

**PROBLEM 7.**
Let $J$ be a fixed ideal in the commutative ring $R$. For each element $a$ of $R$, let $J : a$ be the ideal $\{r \in R \mid ra \in J\}$ in $R$. Prove that every maximal element in the following set of ideals

$$\{J : a \mid a \in R \text{ and } J : a \neq R\}$$

is a prime ideal of $R$.

**PROBLEM 8.**

**PROBLEM 9.**
Let $R$ be the ring $R[x, y]/(x^2 - y^2)$. Identify four distinct prime ideals $P_1$, $P_2$, $Q_1$, and $Q_2$ of $R$ with $P_1 \subset P_2$ and $Q_1 \subset Q_2$.

**PROBLEM 10.**
Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{C}$ with $AB = BA$. Prove that $A$ and $B$ have a common eigenvector. Do $A$ and $B$ have a common eigenvalue?