ANALYSIS QUALIFYING EXAMINATION JANUARY 1998.

Throughout this examination, unless otherwise specified, the terms measurable, a.e., refer to the Lebesgue measure \( m \) on the real line \( \mathbb{R} \), and \( L^p \) of an interval to \( L^p \) of that interval with respect to Lebesgue measure on that interval. Integrals w.r.t. Lebesgue measure will be denoted by \( \int f \, dx \). Problems one through eight are 10 points each. Problem 9 is 20 points.

1. Let \( g \in L_1(\mathbb{R}) \), \( f_n \) measurable functions such that \( f_n \geq g \) and \( f_n \uparrow f \) a.e. Prove that \( \int f_n \, dx \uparrow \int f \, dx \). (Note that by definition \( \int h \, dx = \int h^+ \, dx - \int h^- \, dx \) as long as at least one of the two integrals on the right is finite.)

2. Let \( E, f \) and \( g \) be nondecreasing functions on \([a, b]\) such that \( f + g = F \) and \( F(a) = f(a) = g(a) = 0 \). Prove that \( f \) is absolutely continuous whenever \( F \) is absolutely continuous.

3. Let \( E \subset \mathbb{R} \) be such that there exist \( a \in \mathbb{R} \) and \( \delta > 0 \) such that for all \( |t| < \delta \) we have that \( a - t \in E \) or \( t - a \in E \). Prove that \( m^*(E) \geq \delta \).

4. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that there exists a \( c > 0 \) with \( |f(x) - f(y)| \geq c|x - y| \) for all \( x, y \in \mathbb{R} \).
   a. Show that \( f(\mathbb{R}) \) is closed in \( \mathbb{R} \).
   b. Show that \( f \) is onto.

5. Let \( f \in L_2([0, 1]) \). Let \( g(x, y) = f(x)f(y) \).
   a. Prove that \( g \) is measurable with respect to the product Lebesgue measure.
   b. Prove that \( g \in L_2([0, 1] \times [0, 1]) \) and \( \|g\|_2 = \int |f(x)|^2 \, dx \).

6. Let \( f \) be a measurable function on \( \mathbb{R} \) with \( f \geq 0 \). Prove that there exist measurable sets \( E_n \) and \( \alpha_n \geq 0 \) such that
   \[
   f = \sum_{1}^{\infty} \alpha_n X_{E_n}.
   \]

7. Let \( G \subset \mathbb{C} \) be an open set containing the closed disk \( \overline{D_r(a)} = \{z : |z - a| \leq r\} \). Let \( \langle f_n \rangle \) be a sequence of analytic functions on \( G \) such that \( f_n(z) \to 0 \) uniformly on \( \{z : |z - a| = r\} \). Prove that \( f_n(z) \to 0 \) for all \( z \) in the open disk \( D_r(a) \).

8. Let \( f \) be an analytic function on \( G \), where \( G \) contains the closed unit disk \( \{z : |z| \leq 1\} \) and assume that \( |f(z)| > 2 \) on \( \{z : |z| = 1\} \) and \( f(0) = 1 \). Does \( f \) have to have a zero in the open unit disk?