The field of complex numbers is denoted by $\mathbb{C}$ and the field of rational numbers by $\mathbb{Q}$. The ring of integers is denoted by $\mathbb{Z}$.

1. Let $R$ be a commutative integral domain that satisfies the descending chain condition. Show $R$ is a field.

2. Prove that there are no simple groups of order 231.

3. Let $R$ be the quotient ring $R = \mathbb{Q}[x]/(x^2 + 2x + 2)$, let $\pi : \mathbb{Q}[x] \to R$ be the quotient map, and let $\overline{x} = \pi(x)$ be the image of $x$ under the quotient map.
   (a) Show that $R$ is a field.
   (b) Find the dimension of $R$ as a vector space over $\mathbb{Q}$.
   (c) Express $1/(x^2 - \overline{x}^2 + 4)$ as a polynomial of degree $\leq 1$ in $\overline{x}$.

4. Let $G$ be a finite Abelian group whose order is not divisible by $k^2$ for all integers $k \geq 2$. Prove that $G$ is cyclic.

5. Let $M_{n \times n}$ be the vector space of all $n \times n$ real matrices.
   (a) Show that every $A \in M_{n \times n}$ is similar to its transpose $A^t$.
   (b) Is there a single invertible $S \in M_{n \times n}$ so that $SAS^{-1} = A^t$ for all $A \in M_{n \times n}$?

6. Let $G$ be an Abelian group and let $H$ be a subgroup of $G$ so that the quotient $G/H$ is isomorphic to the additive group of the integers. Then show that there is a subgroup $K$ of $G$ so that $H \cap K = \{1\}$ and $HK = G$.

7. Let $A$ be a $3 \times 3$ matrix over the real numbers and assume that $f(A) = 0$ where $f(x) = x^2(x - 1)^2(x - 2)$. Then give a complete list of the possible values for $\det(A)$.

8. Show that for every polynomial $p(x) \in \mathbb{C}[x]$ of degree $n$ there is polynomial $q(x)$ of degree $\leq n$ so that

\[(x + 1)^n q \left( \frac{x - 1}{x + 1} \right) = p(x).
\]

HINT: Let $P_n$ be the vector space of polynomials of degree $\leq n$ and for $f(x) \in P_n$ define $(Sf)(x) := (x + 1)^n f \left( \frac{(x - 1)(x + 1)}{x + 1} \right)$. Show that $S$ maps $P_n \to P_n$ and is linear. What is the null space of $S$?

9. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree $n \geq 2$ with integer coefficients. Assume that there are $2n + 1$ distinct integers $k_1, k_2, \ldots, k_{2n+1}$ so that each $f(k_i)$ is a prime number. Show $f(x)$ is irreducible over the rational numbers.

10. Let $q$ be an odd number and let $G$ be a finite group of order $2q$. Let $\text{Sym}(G)$ be the permutation group on the set of all elements of $G$ (so that $\text{Sym}(G)$ has order $(2q)!$). Define the right regular representation $\phi : G \to \text{Sym}(G)$ by $\phi(g)\xi = g\xi$.
   (a) Show that $\phi$ is an injective homomorphism from $G$ into $\text{Sym}(G)$. Thus we can view $G$ as a subgroup of $\text{Sym}(G)$.
   (b) Show that $G$ has an element $b$ of order two. Describe how the permutation $\phi(b)$ splits into cycles. Is $\phi(b)$ an even permutation?
   (c) Show that $G$ has a normal subgroup of order $q$. 

- $\xi$