1. Prove that there is no group $G$ so that $G/Z(G) \cong \mathbb{Z}$, where $Z(G)$ is the center of $G$ and $\mathbb{Z}$ is the group of integers under addition.

2. i. Prove that if $A$ and $B$ are $3 \times 3$ matrices over a field $F$, a necessary and sufficient condition that $A$ and $B$ be similar over $F$ is that they have the same characteristic polynomial and the same minimal polynomial.

ii. Give an example to show that (i) is false for $4 \times 4$ matrices.

3. Let $G$ be any group and let $H$ be any subgroup of $G$. Prove that $\text{Aut}(H)$ has a subgroup isomorphic to $N_G(H)/C_G(H)$, where $N_G(H)$ denotes the normalizer of $H$ in $G$, $C_G(H)$ denotes the centralizer of $H$ in $G$, and $\text{Aut}(H)$ denotes the automorphism group of $H$.

4. Let $V$ be the vector space of $n \times n$ matrices over a field $F$. Assume $f$ is a linear functional on $V$ so that $f(AB) = f(BA)$ for all $A, B \in V$, and $f(I) = n$. Prove that $f$ is the trace functional.

5. Let $R$ be a commutative ring. Suppose that $I$ is an ideal of $R$ which is contained in a prime ideal $P$. Prove that the collection of prime ideals containing $I$ and contained in $P$ has a minimal member.

6. Describe, up to isomorphism, all the groups of order 1225.

7. Suppose $N$ is a $4 \times 4$ nilpotent matrix over $F$ with minimal polynomial $x^2$. What are the possible rational canonical forms for $N$?

8. Prove that $y^4 + x^2y + 4xy + x + 4y + 2$ is irreducible in $\mathbb{Q}[x, y]$. Here $\mathbb{Q}$ denotes the field of rational numbers.

9. Let $R$ be a commutative ring with unit. Let $I$ be a prime ideal of $R$ such that $R/I$ satisfies the descending chain condition on ideals. Prove that $R/I$ is a field. [Hint: It is an easier but informative task to prove that every finite integral domain is a field.]

10. Let $A$ and $B$ be $n \times n$ matrices over a field $F$. Prove that $AB$ and $BA$ have the same characteristic polynomial.

11. Suppose that $V$ is an $n$-dimensional vector space over $F$, and $T$ is a linear operator on $V$ which has $n$ distinct characteristic values. Prove that if $S$ is a linear operator on $V$ which commutes with $T$, then $S$ is a polynomial in $T$. 