1. Prove that \( y^3 + x^2 y^2 + x^3 y + x \) is irreducible in \( \mathbb{Z}[x, y] \), where \( \mathbb{Z} \) is the ring of integers.

2. Let \( A \) and \( B \) be \( n \times n \) matrices over a field \( F \). Show that \( AB \) and \( BA \) have exactly the same characteristic values in \( F \).

3. a. Prove that if \( G \) and \( H \) are finite Abelian groups such that \( G \times G \cong H \times H \), then \( G \cong H \).

   b. Provide a counterexample to show that (a.) can be false if \( G \) and \( H \) can be chosen among arbitrary groups.

4. Let \( P \) and \( Q \) be real \( n \times n \) matrices so that \( P + Q = I \) and \( \text{rank}(P) + \text{rank}(Q) = n \). Prove that \( P \) and \( Q \) are projections. (Hint: Show if \( Px = Qy \) for some vectors \( x \) and \( y \), then \( Px = Qy = 0 \).)

5. Prove that if \( G \) is a group of order \( p^2 q \), where \( p \) and \( q \) are prime, then either a \( p \)-Sylow subgroup of \( G \) is normal or a \( q \)-Sylow subgroup of \( G \) is normal.

6. Suppose that \( A \) is an \( n \times n \) real, invertible matrix. Show that \( A^{-1} \) can be expressed as a polynomial in \( A \) with real coefficients and with degree at most \( n - 1 \).

7. Does there exist a polynomial \( f(x) \) in \( \mathbb{R}[x] \) fulfilling all of the following conditions:
   - \( f(x) - 1 \) is in the ideal of \( \mathbb{R}[x] \) generated by \( x^2 + 2x + 1 \), and
   - \( f(x) - 2 \) is in the ideal of \( \mathbb{R}[x] \) generated by \( x - 3 \), and
   - \( f(x) + 1 \) is in the ideal of \( \mathbb{R}[x] \) generated by \( x^2 - 16 \)?

8. Let

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

Determine the rational canonical form of \( A \) and the Jordan canonical form of \( A \).

9. A group \( G \) is said to be solvable provided there is a finite sequence of subgroups \( G = N_0 \supset N_1 \supset \ldots \supset N_m = \{e\} \) such that \( N_i \supset N_{i+1} \) and \( N_i/N_{i+1} \) is Abelian for all \( i < m \). Prove that if \( G \) is a group, \( A \) is a subgroup of \( G \), \( N \) is a normal subgroup of \( G \), and both \( A \) and \( N \) are solvable, then \( AN \) is also solvable.

10. a. Give an example of two \( 4 \times 4 \) nilpotent matrices which have the same minimal polynomial but which are not similar.

   b. Explain why \( 4 \) is the smallest value that can be chosen for the example in part (a.), i.e. if \( n \leq 3 \), any two nilpotent matrices with the same minimal polynomial are similar.