1. Let $E$ and $F$ be disjoint closed subsets of a metric space $(X, d)$. Show that there is a continuous function $f : X \to [0, 1]$ such that

$$f(x) = \begin{cases} 0 & \text{for all } x \in E, \\ 1 & \text{for all } x \in F. \end{cases}$$

2. Let $E$ be a measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Prove that given $\epsilon > 0$, there exists a compact set $F \subseteq E$, such that

$$m(E \sim F) < \epsilon.$$


4. Let $f \in L^1(\mathbb{R})$ and define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(t) = \int_{\mathbb{R}} f(x) e^{itx} \, dx.$$

Prove that if $xf(x) \in L^1(\mathbb{R})$, then $g$ is differentiable on $\mathbb{R}$ and

$$g'(t) = \int_{\mathbb{R}} ix e^{itx} f(x) \, dx.$$

5. Let $\{p_n\}$ be a sequence of $2\pi$-periodic measurable functions on $\mathbb{R}$ satisfying

(a) $p_n(t) \geq 0$ for all $n$ and $t$,

(b) $\int_{-\pi}^{\pi} p_n(t) \, dt = 1$,

(c) For each $\delta > 0$, $\lim_{n \to \infty} \int_{\delta \leq |t| \leq \pi} p_n(t) \, dt = 0$.

For $f$ continuous and $2\pi$-periodic on $\mathbb{R}$, set

$$f_n(x) = \int_{-\pi}^{\pi} p_n(x - t) f(t) \, dt.$$

Prove that $\lim_{n \to \infty} f_n(x) = f(x)$ uniformly on $\mathbb{R}$.
6. Definition. \( f : [a, b] \to \mathbb{R} \) is in \( Lip_1([a, b]) \) if there exists a positive constant \( M \) such that \( |f(x) - f(y)| \leq M|x - y| \) for all \( x, y \in [a, b] \).
   a. Prove that if \( f \in Lip_1([a, b]) \), then \( f \) is absolutely continuous on \([a, b]\).
   b. If \( f \) is absolutely continuous on \([a, b]\), prove that \( f \in Lip_1([a, b]) \) if and only if \( f' \in L^\infty([a, b]) \).

7. Let \( f \) be defined by
   \[
   f(x) = \begin{cases} 
   x^{-1/3}, & 0 < |x| < 1, \\
   0, & x = 0, \text{ or } |x| \geq 1,
   \end{cases}
   \]
   and let \( \{r_n\} \) be an enumeration of the rationals in \( \mathbb{R} \).
   a. Prove that
   \[
   F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n)
   \]
   converges almost everywhere and that the function \( F \) is integrable.
   b. Compute \( \int_{-\infty}^{\infty} F(x) \, dx \).

8. True or False! If the result is true, prove it; if the result is false, provide a counterexample.
   a. If \( f \) is monotone increasing on \([a, b]\) and \( f'(x) = 0 \) a.e. on \([a, b]\), then \( f \) is constant on \([a, b]\).
   b. If \( f \) is monotone on \([a, b]\) and \( f' \) exists a.e. on \([a, b]\), then \( f' \in L^1([a, b]) \).
   c. If \( f \) is differentiable on \((a, b)\), \( c \in (a, b) \), then
   \[
   \lim_{x \to c} f'(x) = f'(c).
   \]
   d. If \( f \) is non-constant and analytic in the open disk \( D = \{z : |z - 3| < 2\} \) and continuous on the closed disk \( \overline{D} = \{z : |z - 3| \leq 2\} \), then the minimum value \( m = \min\{|f(z)| : z \in \overline{D}\} \) cannot be attained by \( |f| \) at any point inside \( D \).


10. a. State the Residue Theorem.
    b. Evaluate \( \int_{\Gamma} \frac{z}{z^2 - 1} \, dz \), where \( \Gamma \) is the ellipse (counterclockwise orientation)
    \[
    \frac{x^2}{3} + 4y^2 = 1.
    \]