1. If $H$ is a subgroup of $G$ such that the product of two right cosets of $H$ is again a right coset of $H$ in $G$, prove that $H$ is normal in $G$.

2. The multiplicative group of $n \times n$ invertible real matrices is denoted $GL(n, \mathbb{R})$. Let

$$O_n = \{ A \in GL(n, \mathbb{R}) \mid A^t A = I \}$$

$$SO_n = \{ A \in O_n \mid \det A = 1 \}$$

a. Prove that $SO_n \triangleleft O_n \triangleleft GL(n, \mathbb{R})$ and compute $[O_n : SO_n]$.

b. Prove or disprove that $O_2 = SO_2 \times \{ \pm I \}$ and $O_3 = SO_3 \times \{ \pm I \}$.

3. Show that a finite abelian group fails to be cyclic if and only if it has a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime number $p$.

4. Let $R$ be a UFD. Prove that $f(x, y, z) = x^5 y^3 + x^4 z^3 + x^3 yz^2 + y^2 z$ is irreducible in $R[x, y, z]$.

5. Let $V$ be the vector space of all $n \times n$ matrices over a field $F$, and let $B$ be a fixed $n \times n$ matrix that is not of form $cI$. Define a linear operator $T$ on $V$ by $T(A) = AB - BA$. Exhibit a non-zero element of the kernel of the transpose of $T$.

6. Suppose $G$ is a group of order 66.
   a. Prove that $G$ is solvable, i.e., there is a chain of subgroups $G \supset G_1 \supset G_2 \supset \cdots \supset G_k = \{ e \}$ such that each factor group $G_i/G_{i+1}$ is abelian.
   b. Prove that $G$ has a normal subgroup of order 3.