PH.D. QUALIFYING EXAMINATION  
ALGEBRA PORTION  
AUGUST 1991

1. Let $R$ be a ring and $\text{Aut}(R)$ be the group of ring automorphisms of $R$.
   (a) Show that $\text{Aut}(R) = \{1\}$.
   (b) Find $\text{Aut}(R[x])$. (Hint: It is isomorphic to a subgroup of $\text{GL}(2, \mathbb{R})$.)

2. For each field $F$ given below, factor $x^{31} - 1 \in F[x]$ into a product of irreducible polynomials and justify your answer.
   (a) $F$ is the field of complex numbers.
   (b) $F$ is the field of rational numbers.
   (c) $F$ is the field of 31 elements.
   (d) $F$ is the field of 32 elements.

3. Let $A$ and $B$ be $n \times n$ matrices with entries from $\mathbb{R}$. Suppose that $A$ and $B$ are similar over $\mathbb{C}$. Prove that they are similar over $\mathbb{R}$.

4. Let $A$ be an $n \times n$ matrix with entries in a field $F$. Suppose that $A^2 = A$. Prove that the rank of $A$ is equal to the trace of $A$.

5. Let $D$ be a commutative domain and $F$ be its field of fractions. The domain $D$ is said to be neat if whenever $f(x)$ and $g(x)$ are monic polynomials in $F[x]$ with $f(x)g(x) \in D[x]$, then both $f(x)$ and $g(x)$ are in $D[x]$.
   (a) Prove that if $D$ is a UFD, then $D$ is neat.
   (b) Give an example of a domain $D$ which is not neat.