# 1. Suppose that $T: V \rightarrow W$ is an injective linear transformation of vector spaces over a field $F$. Prove that $T^*: W^* \rightarrow V^*$ is surjective. (Recall that $V^* = L(V, F)$ is the vector space of linear transformations from $V$ to $F$, and $T^*(f) = f \circ T$ for all $f \in W^*$.)

# 2. Let $R$ be the ring of $2 \times 2$ matrices over the field of complex numbers. Find two left ideals $I_1$ and $I_2$ of $R$ such that
   (a) $I_1$ and $I_2$ are isomorphic as left $R-$ modules, and
   (b) $I_1 \neq I_2$.
   Prove that $I_1$ and $I_2$ satisfy (a) and (b).

# 3. If $M$ is the $n \times n$ matrix
   \[
   M = \begin{bmatrix}
   x & a & a & \cdots & a \\
   a & x & a & \cdots & a \\
   & & & & \\
   a & a & \cdots & a & x
   \end{bmatrix},
   \]
   then prove that $\det M = (x + (n-1)a)(x - a)^{n-1}$.

# 4. Let $F$ be a field and let $G$ be a finite subgroup of the multiplicative group $F \setminus \{0\}$. Prove that $G$ is a cyclic group.
5. If $G$ is a group of order $p^n$ for some prime integer $p$, and $H$ is a subgroup of $G$ with $H \neq G$, then prove that there exists a subgroup $N$ of $G$ such that $H \subseteq N$, $H \neq N$, but $H$ is normal in $N$. (You are expected to write a complete proof. The answer “this is a well known theorem” is not acceptable.) (W.1)

6. Suppose that $T$ is a linear operator on a finite-dimensional vector space $V$ over a field $F$. Prove that $T$ has a cyclic vector if and only if

$$\{U \in L(V, V) : TU = UT\} = \{f(T) : f \in F[x]\}.$$  

7. Let $I$ be an ideal in a commutative ring $R$ and let $\mathcal{S}$ be the set of ideals of $R$ defined by the property that $J \in \mathcal{S}$ if and only if there exists an element $a$ of $R$ such that $a \notin I$ and $J = \{r \in R | ra \in I\}$. Prove that every maximal element of $\mathcal{S}$ is a prime ideal in $R$.

8. Let $F$ be a field. Let $f_1, \ldots, f_r$ be polynomials in the polynomial ring $F[x]$.

(a) Fill in the blank. The natural map

$$F[x] \to \frac{F[x]}{(f_1)} \oplus \cdots \oplus \frac{F[x]}{(f_r)}$$

is onto if and only if

(b) Prove your answer to (a). (You are expected to write a complete proof. The answer “this is a well known theorem” is not acceptable.)
Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_1, -2x_2 - x_3 - 4x_4, 4x_2 + x_3).$$

\(\checkmark\) (a) Compute the characteristic polynomial of $T$.
\(\checkmark\) (b) Compute the minimal polynomial of $T$.
\(\checkmark\) (c) The vector space $\mathbb{R}^4$ is the direct sum of two proper $T$-invariant subspaces. Exhibit a basis for one of these subspaces.