1. Let $G = G(n, F)$ be the multiplicative group of $n \times n$ invertible matrices over the field $F$, and let $S = \{A \in G \mid \det A = 1\}$. Prove that $S$ is a normal subgroup of $G$ and that $G/S$ is isomorphic to $F^\times$, the multiplicative group of nonzero elements of $F$.

2. Let $V$ and $W$ be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Prove that there exists a basis $A$ of $V$ and a basis $B$ of $W$ such that the matrix $[T]_{A,B}$ has the block form \[
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\].

3. Show that every finite group with more than two elements has a nontrivial automorphism. (Hint: consider the abelian and non-abelian cases separately.)

4. Prove that there is no simple group of order 56.

5. Let $V$ be a finite-dimensional vector space and let $T$ be a diagonalizable linear operator on $V$. Prove that if $W$ is a $T$-invariant subspace of $V$ then the restriction of $T$ to $W$ is diagonalizable.

6. Let $R = Z[X]$. Give three prime ideals in $R$ that contain the ideal $(6, 2X)$, and prove that your ideals are prime.

7. Let $T$ be a linear operator on a finite-dimensional vector space $V$. Show that if $T$ has no cyclic vector, then there exists an operator $U$ that commutes with $T$ but is not a polynomial in $T$.

8. Let $f(x) = x^4 + 2x^3 + 10x^2 + 16x + 16$ and $g(x) = x^4 + 2x^3 + 3x^2 + 2x + 2$ in the ring $C[x]$.
   a) Compute the greatest common divisor of $f$ and $g$, i.e. the monic generator of the ideal $(f, g)$.
   b) If $f$ is the characteristic polynomial of a certain complex matrix $A$, decide whether or not $g(A)$ is singular.

9. Let $R$ be a commutative ring with identity and let $I$ and $J$ be ideals of $R$. Define $IJ$ to be the ideal generated by all products $xy$ with $x \in I$ and $y \in J$; that is, $IJ$ is the set of all finite sums of such products.
   a) Prove that $IJ \subseteq I \cap J$.
   b) Prove that $IJ = I \cap J$ if $R$ is a principal ideal domain and $I + J = R$.
   c) We say that $R$ has the descending chain condition (DCC) if given any chain of ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ there is an integer $k$ such that $I_k = I_{k+1} = I_{k+2} = \cdots$. If $R$ has DCC, prove that $R$ has only finitely many maximal ideals. (Hint: If $M_1, M_2, \ldots$ are distinct maximal ideals, consider the ideals $I_j = M_1 \cap \cdots \cap M_j$.)