

1 Complex Analysis

(92 Aug.) Suppose that $f(z)$ is an analytic function on the open disk $D_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$. Prove: If $f(z)$ is not identically zero, then there exists $\delta > 0$ such that $f(z) \neq 0$ for all $z \in D_\delta(a) \setminus \{a\}$.

(93 Jan.) Find all entire functions f so that $f(2z) = -if(z)$ for all $z \in \mathbb{C}$.

(93 Jan.) Let μ be a finite measure on a compact set $K \subset \mathbb{C}$. Define F by

$$F(z) = \int \frac{1}{w - z} d\mu(w).$$

Prove F is analytic with derivative

$$F'(z) = \int \frac{1}{(w - z)^2} d\mu(w)$$

on the complement of K in \mathbb{C} .

(94 Jan.) Let $f(z)$ be analytic on a domain Ω , except for poles in Ω : Prove that the only singularities of

$$g(z) = \frac{f'(z)}{f(z) - A}$$

are simple poles at all the poles of f and all the points $z \in \Omega$ such that $f(z) = A$.

(94 Jan.) Compute

$$\oint_C \frac{z}{(z - 1)^2(z - 2)^2} dz,$$

where C is the circle $|z - 2| = \frac{1}{2}$, traversed counterclockwise.

(94 Jan.) Let $\{f_n\}$ be a uniformly bounded sequence of analytic functions on Ω such that $f_n(z)$ converges pointwise for all $z \in \Omega$. Prove that $\{f_n\}$ converges uniformly on every compact subset of Ω .

(94 Aug.) Let $G \subset \mathbb{C}$ be a region and let $\langle f_n \rangle$ be a sequence of holomorphic functions on G which converges uniformly on every compact subset of G to a function f . Prove that f is holomorphic on G .

(94 Aug.) Let f be an entire function on \mathbb{C} and assume that $|f(z)| \leq A|z|^k + B$ for some constants A, B , integer k and all $z \in \mathbb{C}$. Prove that f is a polynomial.

(95 Jan.) Use the Residue Theorem to evaluate

$$\int_0^{\infty} \frac{\cos mx}{x^2 + 1} dx$$

where $m > 0$

(95 Jan. 99 Jan.)

(a) State the Cauchy's Integral Formula.

(b) Prove the Liouville's Theorem.

(95 Aug.) State and prove the Fundamental Theorem of Algebra.

(95 Aug.) Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 2x + 2)} dx.$$

(95 Aug.) Let

(1) $z_0 \in \mathbb{C}$

(2) $0 < r < R$

(3) f and g be analytic functions on $D(z_0, R) \equiv \{z \in \mathbb{C} : |z - z_0| < R\}$

(4) $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$

(5) $|g(z)| < |f(z)|$ for all $z \in \gamma^* \equiv \{\gamma(t) \in \mathbb{C} : 0 \leq t \leq 2\pi\}$.

Show that the number of zeros of f inside of γ is equal to the number of zeros of $f + g$ inside of γ .

(96 Jan.) Use the Residue Theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} dx.$$

(96 Jan.) Let a be an isolated singularity of the function f and $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ be its Laurent Expansion in $D'_\delta(a)$ for some $\delta > 0$, where

$$D_\delta(a) = \{z \in \mathbb{C} : |z - a| \leq \delta\} \text{ and } D'_\delta(a) = D_\delta(a) \setminus \{a\}.$$

- (a) Fill in the blanks with the proper restrictions on a_n 's:
- (i) $z = a$ is a removable singularity if and only if
 - (ii) $z = a$ is a pole or order m if and only if
 - (iii) $z = a$ is a essential singularity if and only if
- (b) Prove the Casorati-Weierstrass Theorem; namely, show that if f has an essential singularity at $z = a$ and $\delta > 0$, then the set:

$$f(D'_\delta(a)) \equiv \{f(z) : z \in D'_\delta(a)\}$$

is dense in \mathbb{C} .

(96 Jan.) State Rouché's Theorem and then use Rouché's Theorem to prove the Fundamental Theorem of Algebra.

(96 Aug.) Evaluate

$$\int_0^{\infty} \frac{x^2}{1+x^4} dx.$$

(96 Aug.) **EITHER:** Let f be an entire function. Prove carefully that f is a polynomial of degree n if and only if there exists a non-zero $\alpha \in \mathbb{C}$ such that $\lim_{z \rightarrow 0} z^n f(1/z) = \alpha$.

OR:(i) What is the meant by the "principle of the isolated zeros"?

(ii) Let f be holomorphic and not identically zero on the region Ω . Prove that $Z(f) = \{z \in \Omega : f(z) = 0\}$ is countable and that $Z(f) \cap K$ is finite for every compact set K wholly contained in Ω .

(97 Jan.) Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx.$$

(97 Jan.) In this question Δ denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Suppose that $f : \Delta \rightarrow \Delta$ is holomorphic and that $f(0) = 0$.

- (i) Prove that $g(z) = f(z)/z$ has a removable singularity at $z = 0$ and that $|g(z)| \leq 1$ ($z \in \Delta \setminus \{0\}$).
- (ii) Deduce that either $|f'(0)| < 1$ or $f(z) = cz$ ($z \in \Delta$) for some complex number c with $c = 1$.

(97 Aug.) Prove that a bounded entire function is constant and use this to prove the fundamental theorem of algebra.

(97 Aug.) Let γ be the curve in Figure 1. Then compute

$$\int_{\gamma} \frac{1 + z + z^3}{z(z-1)(z-i)} dz.$$

(98 Jan.) Let $G \subset \mathbb{C}$ be an open set containing the closed disk $\overline{D_r(a)} = \{z : |z - a| \leq r\}$. Let $\langle f_n \rangle$ be a sequence of analytic functions on G such that $f_n(z) \rightarrow 0$ uniformly on $\{z : |z - a| = r\}$. Prove that $f_n(z) \rightarrow 0$ for all z in the open disk $D_r(a)$.

(98 Jan.) Let f be an analytic function on G , where G contains the closed unit disk $\{z : |z| \leq 1\}$ and assume that $|f(z)| > 2$ on $\{z : |z| = 1\}$ and $f(0) = 1$. Does f have to have zero in the open unit disk?

(98 Aug.) Let G be an open connected subset of \mathbb{C} . Let f be a continuous function on G such that e^f is a constant on G . say $e^{f(z)} = c$ for each $z \in G$ for some $c \in \mathbb{C}$. Show that f is constant on G .

(98 Aug.) Show Liouville's Theorem: If f is an holomorphic bounded function on \mathbb{C} , then f is constant on \mathbb{C} . Hint: it suffices to show that f' is zero on \mathbb{C} .

(99 Jan.) Prove the Fundamental Theorem of Algebra using Liouville's Theorem.

(99 Aug.)

- (a) For each positive integer n , show that $F_n(z) = \int_{\frac{1}{n}}^1 e^{-t} t^{z-1} dt$ is analytic in $\operatorname{Re} z > 0$, where $t^z = e^{z \ln t}$
- (b) Show that for every $\sigma > 0$, $F_n(z)$ converges uniformly to an analytic function $F(z)$ on $\operatorname{Re} z \geq \sigma$.

(00 Jan.) Suppose that g is an entire function satisfying $|g(z)| \geq M|z|^N$ for all sufficiently large z , where N is a positive integer and $M > 0$. Prove that g is a polynomial of degree at least N .

(00 Jan.)

(a) Suppose that f is analytic on the disk $D = \{|z| < 2\}$ and has no zeros on the unit circle $C = \{|z| = 1\}$. Using the Residue Theorem, show that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N,$$

where C is positively oriented and N is the number of zero of f inside C counted according to multiplicity.

(b) Now suppose that g is also analytic on D and that $(1-t)f(z) + tg(z) \neq 0$ for all $t \in [0, 1]$ and $z \in \mathbb{C}$. Deduce that

$$\frac{1}{2\pi i} \int_C \frac{g'(z)}{g(z)} dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz.$$

(00 Aug.)

(a) State and prove Liouville's theorem.

(b) Suppose f is an entire function and $|f(z)| \leq Me^{\operatorname{Re}z}$ for every z from the complex plane, where M is a constant. Prove that there exists a constant C such that $f(z) = Ce^z$.

(01 Aug.) Prove Liouville's Theorem.

(01 Aug.) Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^2}$$

by contour integration.

(02 Aug.) Let $\Omega \subset \mathbb{C}$ be an open set containing the unit disk $\{z : |z| \leq 1\}$ and let $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| > |f(0)|$ for all $|z| = 1$. Prove that f has a zero in $|z| < 1$.

(02 Aug.) Let $f, g : \{z : |z| < 1\} \rightarrow \mathbb{C}$ be holomorphic functions such that $|f(z)| = |g(z)|$ for all $|z| < 1$. Prove that every zero of g is also a zero of f of the same multiplicity and thus $f = \lambda g$ for some λ with modulus one.

(03 Jan.) Let f be a non-constant entire function such that $f(\mathbb{R}) \subset \mathbb{R}^+$. Prove that all real zeros of f have even order.

(03 Jan.) Let $f : \{z : |z| < 1\} \rightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| < \frac{1}{|z|}$ for all $z \neq 0$. Prove that $|f(z)| \leq 1$ for all $|z| < 1$.

(03 Jan.) Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx.$$

(03 Aug.) Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Compute

$$\int_{\gamma} \left(\frac{z}{z-1}\right)^n dz$$

for all positive integer n .

(03 Aug.) Let Ω be a region in \mathbb{C} and $f, g : \Omega \rightarrow \mathbb{C}$ be analytic functions such that $f(z)g(z) = 0$ for all $z \in \Omega$. Then prove that $f \equiv 0$ or $g \equiv 0$.

(03 Aug.) Evaluate

$$\int_0^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx.$$

(04 Jan.) Let $G \subset \mathbb{C}$ be a region and let $f : G \rightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| = c$ for all $z \in G$. Prove that f is constant on G .

(04 Jan.) Let f be a holomorphic function defined in a neighborhood of the origin and let $f'(0) \neq 0$.

(a) Show that there exists $r > 0$ such that the unique solution of the equation $f(z) = f(0)$ in the disc $|z| < r$ is $z = 0$.

(b) Prove that if $r > 0$ is sufficiently small, then

$$\frac{1}{f'(0)} = \frac{1}{2\pi i} \int_{|z|=r} \frac{1}{f(z) - f(0)} dz,$$

where the circle $|z| = r$ is traversed counterclockwise.

(04 Jan.) Let $f : \{z : |z| < 1\} \rightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq 1$ for all $|z| < 1$, $f(0) = 0$, and $|f(\frac{1}{4})| = \frac{1}{4}$. Prove that there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cz$ for all $|z| < 1$.

(04 Aug.) Suppose that $f(z)$ is analytic on a convex domain U and $\gamma(t)$ $0 \leq t \leq 1$ is a positively-oriented simple closed smooth parameterized curve contained in U .

(a) Write down an integral on $[0, 1]$ for the length $L(\gamma)$ of γ .

(b) Write down Cauchy's Integral Formula for $f(z)$ for a point z inside γ^* .

(c) Deduce that

$$|f(z)| \leq \frac{ML(\gamma)}{2\pi \text{dist}(z, \gamma^*)}.$$

where M denotes the maximum value of $|f(z)|$ on γ^* and $\text{dist}(z, \gamma^*)$ denotes the distance from z to γ^* .

(04 Aug.)

(a) Suppose that $f : \Delta \rightarrow \Delta$ is an analytic mapping from the open unit disk Δ to itself such that $f(0) = 0$. By considering $f(z)/z$, show that $|f'(0)| \leq 1$.

(b) Deduce that if $h : H \rightarrow \Delta$ is an analytic mapping from the upper half-plane $H = \{z = x + iy : y > 0\}$ into Δ , with $h(i) = 0$, then $|h'(i)| \leq 1/2$. (Hint: Consider $g(z) = (z - i)/(z + i)$.)

(05 Aug.) Let C be the circle $|z| = 1$ traversed once counterclockwise. For $a > 1$, show that

$$\int_C \frac{dz}{z^2 + 2az + 1} = \frac{\pi i}{\sqrt{a^2 - 1}}$$

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- (05 Aug.) Suppose that $f(z)$ is an entire function which satisfies $|f(z)| \leq C(1 + |z|^N)$, where $C > 0$ and $N \in \mathbb{N}$. Prove that $f(z)$ is a polynomial of degree at most N .
- (06 Jan.) Suppose that f is analytic on $\{z : |z| < 2\}$, that $f(1/3) = 1 + i$, and that $|f(z)| > 2$ if $|z| = 1$. Prove carefully that f has a zero in $\{z : |z| < 1\}$.
- (06 Jan.) Suppose that f is analytic on \mathbb{C} except for a finite number of singularity and that $zf(z) \rightarrow 0$ as $z \rightarrow \infty$. Prove carefully that there exists $M > 0$ such that $|z^2f(z)| \leq M$ for all z such that $|z| \geq M$. (Hint: Consider $f(1/z)$)