January/February Solutions

Problem

Let

$$b(x) = (x-a)^2, \quad 0 \le x \le 1,\ 0 \le a \le 1.$$

Denote the mean value of the function $b(x)$ on the interval from s to t by the formula

$$M(s,t) = \frac{1}{t-s}\int_s^t b(x)\ dx, \quad 0 \le s \le 1,\ 0 \le t \le 1.$$

Prove that

$$M(s,t) \ge \frac{b(t)}{4} \text{ for all } 0 \le s \le 1,\ 0 \le t \le 1.$$

Solution

After integration, we get

$$\begin{aligned} M(s,t) &= \frac{(t-a)^3 - (s-a)^3}{3(t-s)} \\ &= \frac{[(t-a)-(s-a)][(t-a)^2 + (t-a)(s-a) + (s-a)^2]}{3(t-s)} \\ &= \frac{(t-a)^2 + (t-a)(s-a) + (s-a)^2}{3} \end{aligned}$$

Completing the square, we have

$$\begin{aligned} M(s,t) &= \frac{1}{3}\left(\frac{(t-a)^2}{4} + (t-a)(s-a) + (s-a)^2 + \frac{3(t-a)^2}{4}\right) \\ &= \frac{1}{3}\left(\left(\frac{t-a}{2}\right) + (s-a)\right)^2 + \frac{(t-a)^2}{4} \ge \frac{(t-a)^2}{4}. \end{aligned}$$

Correct Solutions

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