February/March Solutions

Problem

Solve the equation

$$(\ln x)^2 - 2.5(\ln x)(\ln(4x-5)) + (\ln(4x-5))^2 = 0,$$

where x and all expressions in the equation are real.

Solution

Notice that x must be greater than $\frac{5}{4}$ for $\ln(4x-5)$ to be real, so $\ln x$ is never 0. Divide the equation by $(\ln x)^2$ to get

$$1 - 2.5\left(\frac{\ln(4x-5)}{\ln x}\right) + \left(\frac{\ln(4x-5)}{\ln x}\right)^2 = 0$$

Denote $u = \frac{\ln(4x-5)}{\ln x}$, so the equation becomes

$$1 - 2.5u + u^2 = 0,$$

which factors as

$$(2-u)(0.5-u) = 0,$$

so $u = 2$ or $u = 0.5$. If

$$\frac{\ln(4x-5)}{\ln x} = 2,$$

then $4x - 5 = x^2$, but applying the quadratic formula to $0 = x^2 - 4x + 5$ shows that it has no real roots. If

$$\frac{\ln(4x-5)}{\ln x} = 0.5,$$

then $4x - 5 = x^{0.5}$ or $16x^2 - 40x + 25 = x$, so

$$16x^2 - 41x + 25 = (16x - 25)(x - 1) = 0.$$

The solution $x = 1$ is not valid, since $x > \frac{5}{4}$, so the only solution is $x = \frac{25}{16}$.

Correct Solutions

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