On generalizations of the Littlewood-Paley inequalities to domains in $\mathbb{R}^n$ $(n \geq 2)$ [1]

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ABSTRACT. For the unit disc $\mathbb{D}$ in $\mathbb{C}$, the harmonic Hardy spaces $\mathcal{H}^p$, $1 \leq p < \infty$, are defined as the set of harmonic functions h on $\mathbb{D}$ satisfying

$$\|h\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta < \infty.$$

The classical Littlewood-Paley inequalities for harmonic functions [**5**] in $\mathbb{D}$ are as follows: Let h be harmonic on $\mathbb{D}$. Then there exist positive constants C_1, C_2, independent of h, such that

(a) for $1 < p \leq 2$,

$$\|h\|_p^p \leq C_1 \left[|h(0)|^p + \iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \right]. \tag{0.1}$$

For analytic functions inequality (0.1) is valid for all p, $0 < p \leq 2$ ([**4**]).

(b) For $p \geq 2$, if $h \in \mathcal{H}^p$, then

$$\iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \leq C_2 \|h\|_p^p. \tag{0.2}$$

In the paper we consider extensions of these inequalities to Hardy-type spaces of harmonic functions on domains $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$. For $1 \leq p < \infty$, $\mathcal{H}^p$ denotes the set of harmonic functions h on Ω for which $|h|^p$ has a harmonic majorant on Ω. A suitable norm on $\mathcal{H}^p$ can be defined by $N_p(h) = (H_{|h|^p}(t_o))^{1/p}$, where $H_{|h|^p}$ is the least harmonic majorant of $|h|^p$ and t_o is a fixed point in Ω. It is proved that the analogues of (0.1) and (0.2) are valid on those domains Ω in $\mathbb{R}^n$ for which the Green's function G satisfies $G(t_o, x) \approx \delta(x)$ for all $x \in \Omega \setminus B(t_o, \frac{1}{2}\delta(t_o))$, where $\delta(x)$ denotes the distance from x to the boundary of Ω. We also prove a generalization of the Littlewood-Paley inequalities for bounded Lipschitz domains, as well as a generalization of inequality (0.1) valid for all p, $0 < p \leq 1$, for the case where Ω is a bounded domain with $C^{1,1}$ boundary.

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1. Introduction

For the unit disc $\mathbb{D}$ in $\mathbb{C}$, the harmonic Hardy spaces $\mathcal{H}^p$, $1 \le p < \infty$, are defined as the set of harmonic functions h on $\mathbb{D}$ satisfying

$$\|h\|_p^p = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^p d\theta < \infty.$$

The classical Littlewood-Paley inequalities for harmonic functions [**5**] in $\mathbb{D}$ are as follows: Let h be harmonic on $\mathbb{D}$. Then there exist positive constants C_1, C_2, independent of h, such that
(a) for $1 < p \le 2$,

$$\|h\|_p^p \le C_1 \left[|h(0)|^p + \iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \right]. \tag{1.1}$$

For analytic functions inequality (1.1) is valid for all p, $0 < p \le 2$ [**4**].
(b) For $p \ge 2$, if $h \in \mathcal{H}^p$, then

$$\iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \le C_2 \|h\|_p^p. \tag{1.2}$$

In 1956 T. M. Flett [**4**] proved that for analytic functions inequality (1.1) is valid for all p, $0 < p \le 2$. Hence if $u = \operatorname{Re} h$, h analytic, then since $|\nabla u| = |h'|$ it immediately follows that inequality (1.1) also holds for harmonic functions in $\mathbb{D}$ for all p, $0 < p \le 2$. A short proof of the Littlewood-Paley inequalities for harmonic functions in $\mathbb{D}$ valid for all p, $0 < p < \infty$ has also been given recently by Pavlović in [**8**]. The Littlewood-Paley inequalities are also known to be valid for harmonic functions in the unit ball in $\mathbb{R}^n$. In fact Stević [**11**] has recently proved that for $n \ge 3$, inequality (1.1) is valid for all $p \in [\frac{n-2}{n-1}, 1]$.

In the paper we extend the Littlewood-Paley inequalities to harmonic functions on domains $\Omega \subsetneq \mathbb{R}^n$, $n \ge 2$, with Green function G satisfying the following conditions: For fixed $t_o \in \Omega$, there exist constants C_1 and C_2, depending only on t_o, such that

$$C_1 \delta(x) \le G(t_o, x) \qquad \text{for all } x \in \Omega, \text{ and} \tag{1.3}$$

$$G(t_0, x) \le C_2 \delta(x) \qquad \text{for all } x \in \Omega \setminus B(t_o, \tfrac{1}{2}\delta(t_o)), \tag{1.4}$$

where $\delta(x)$ denotes the distance from x to the boundary of Ω.[2]

Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \ge 2$. As in [**10**], for $1 \le p < \infty$, we denote by $\mathcal{H}^p(\Omega)$ the set of harmonic functions h on Ω for which $|h|^p$ has a harmonic majorant on Ω. For $h \in \mathcal{H}^p$ we denote the least harmonic majorant of $|h|^p$ by $H_{|h|^p}$, and for fixed $t_o \in \Omega$ we set

$$N_p(h) = \left(H_{|h|^p}(t_o) \right)^{1/p} \tag{1.5}$$

Then N_p is a norm on $\mathcal{H}^p$. It is known that this norm is equal to

$$\lim_{n\to\infty} \left(\int_{\partial\Omega_n} |h(t)|^p d\omega_n^{t_o}(t) \right)^{1/p},$$

[2]If the boundary of Ω is C^2 or $C^{1,1}$, then the inequalities can be established by comparing the Green function G to the Green function of balls that are internally and externally tangent to the boundary of Ω. By the results of Widman [**15**], the inequalities are also valid for domains with $C^{1,\alpha}$ or Liapunov-Dini boundaries.

where $\{\Omega_n\}$ is a regular exhaustion of Ω and $\omega_n^{t_o}$ is the harmonic measure on $\partial\Omega_n$ with respect to the point t_o. Here we assume that $t_o \in \Omega_n$ for all n.

In the paper we prove the following theorems.

THEOREM 1. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with $\Omega \subsetneq \mathbb{R}^n$, and let h be a real-valued harmonic function on Ω and $\gamma \in \mathbb{R}$. Then there exist positive constants C_1, C_2, independent of h, such that*
(a) *for $1 < p \leq 2$*

$$\int_\Omega \delta(x)^\gamma |h(x)|^{p-2} |\nabla h(x)|^2 dx \leq C_1 \int_\Omega \delta(x)^{\gamma+p-2} |\nabla h(x)|^p dx.$$

(b) *For $p \geq 2$,*

$$\int_\Omega \delta(x)^\gamma |h(x)|^{p-2} |\nabla h(x)|^2 dx \geq C_2 \int_\Omega \delta(x)^{\gamma+p-2} |\nabla h(x)|^p dx.$$

As a consequence of Theorem 1 we obtain the following generalization of the Littlewood-Paley inequalities.

THEOREM 2. *Let Ω be a domain in $\mathbb{R}^n$, $\Omega \subsetneq \mathbb{R}^n$, with Green function G satisfying (1.3) and (1.4), and let $t_o \in \Omega$ be fixed. There exist constants C_1 and C_2, depending only on Ω and p, such that for all real-valued harmonic functions h on Ω*
(a) *for $1 < p \leq 2$,*

$$N_p(h) \leq C_1 \left[|h(t_o)| + \left(\int_\Omega \delta(x)^{p-1} |\nabla h(x)|^p dx \right)^{1/p} \right].$$

(b) *If $h \in \mathcal{H}^p(\Omega)$, $2 \leq p < \infty$, then*

$$\left(|h(t_o)|^p + \int_\Omega \delta(x)^{p-1} |\nabla h(x)|^p dx \right)^{1/p} \leq C_2 N_p(h).$$

In Section 4 we prove a generalization of the Littlewood-Paley inequalities for bounded Lipschitz domains in $\mathbb{R}^n$.

If Ω is a bounded domain with $C^{1,1}$ boundary, then the norm $N_p(h)$ can be replaced by $\|h\|_p$, where for $1 \leq p < \infty$,

$$\|h\|_p = \sup_{0<r<r_o} \left[\int_{\partial\Omega} |f(t - r\mathbf{n}_t)|^p dS(t) \right]^{1/p}.$$

In the above, for each $t \in \partial\Omega$, $\mathbf{n}_t$ denotes the unit outward normal at t, and dS denotes surface measure on $\partial\Omega$ (see [**13**] for details).

In Section 5 of the paper we will prove the following generalization of the Littlewood-Paley inequality valid for all p, $0 < p \leq 1$.

THEOREM 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary, and let $t_o \in \Omega$ be fixed. Then there exists $\rho_o > 0$ and a constant $C > 0$, depending only on Ω and p, such that for all p, $0 < p \leq 1$, and all harmonic functions h on Ω,*

$$\int_{\partial\Omega} |h(t - r\boldsymbol{n}_t)|^p dS(t) \leq C \left[|h(t_o)|^p + \int_\Omega \delta(y)^{p-1} |\nabla h(y)|^p dy \right],$$

for all r, $0 < r < \rho_o$.

2. Preliminaries

Our setting throughout the paper is $\mathbb{R}^n$, $n \geq 2$, the points of which are denoted by $x = (x_1, ..., x_n)$ with euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. For $r > 0$ and $x \in \mathbb{R}^n$, set $B_r(x) = B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ and $S_r(x) = S(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\}$. For convenience we denote the ball $B(0, \rho)$ by B_ρ, and the unit sphere $S_1(0)$ by S. Lebesgue measure in $\mathbb{R}^n$ will be denoted by $d\lambda$ or simply dx, and the normalized surface measure on S by $d\sigma$. The volume of the unit ball B_1 in $\mathbb{R}^n$ will be denoted by ω_n. Finally, for a real (or complex) valued C^1 function f, the gradient of f is denoted by ∇f, and if f is C^2, the Laplacian Δf of f is given by

$$\Delta f = \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}.$$

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$, with $\Omega \subsetneq \mathbb{R}^n$. For $x \in \Omega$ set

$$B(x) = B(x, \tfrac{1}{2}\delta(x)) = \left\{y \in \Omega : |y - x| < \tfrac{1}{2}\delta(x)\right\}. \tag{2.1}$$

Then for all $y \in B(x)$ we have

$$\tfrac{1}{2}\delta(x) \leq \delta(y) \leq \tfrac{3}{2}\delta(x). \tag{2.2}$$

For the proofs of Theorems 1 and 2 we require several preliminary lemmas.

Lemma 1. *For $f \in L^1(\Omega)$ and $\gamma \in \mathbb{R}$,*

$$\int_\Omega \delta(x)^\gamma |f(x)|\, dx \approx^{3} \int_\Omega \delta(w)^{\gamma - n} \left[\int_{B(w)} |f(x)|\, dx\right] dw.$$

Proof. For $E \subset \Omega$, let χ_E denote the characteristic function of E. Thus

$$\int_\Omega \delta(y)^{\gamma - n} \left[\int_{B(y)} |f(x)|\, dx\right] dy = \int_\Omega \int_\Omega \delta(y)^{\gamma - n} \chi_{B(y)}(x) |f(x)|\, dx\, dw.$$

But by (2.2), $\chi_{B(y)}(x) \leq \chi_{2B(x)}(y)$. Thus

$$\begin{aligned}
&\int_\Omega \int_\Omega \delta(x)^{\gamma - n} \chi_{B(y)}(x) |f(x)| dx\, dy \\
&\qquad \leq C \int_\Omega \int_\Omega \delta(x)^{\gamma - n} \chi_{2B(x)}(y) |f(x)|\, dy\, dx \\
&\qquad \leq C \int_\Omega \delta(x)^\gamma |f(x)|\, dx.
\end{aligned}$$

The reverse inequality follows likewise. □

Lemma 2. *For $u \in C^2(\overline{B_\rho})$, $\rho > 0$,*

$$\int_S u(\rho\zeta)\, d\sigma(\zeta) = u(0) + \int_{B_\rho} \Delta u(x) G_\rho(x)\, dx,$$

where

$$G_\rho(x) = \begin{cases} \dfrac{1}{n(n-2)\omega_n} \left[\dfrac{1}{|x|^{n-2}} - \dfrac{1}{\rho^{n-2}}\right], & 0 < |x| \leq \rho, \quad n \geq 3, \\ \dfrac{1}{2\pi} \log \dfrac{\rho}{|x|}, & 0 < |x| \leq \rho, \quad n = 2, \end{cases} \tag{2.3}$$

[3] The notation $A \approx B$ means that there exist constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$.

is the Green function of $B(0,\rho)$ *with singularity at* 0.

PROOF. The proof is an immediate consequence of Green's formula and hence is omitted. □

LEMMA 3. *Let* u *be a real-valued harmonic on* $\overline{B(0,\rho)}$, $\rho > 0$.
(a) *For* $1 < p \leq 2$,

$$\int_{B_{\rho/4}} |u(x)|^{p-2}|\nabla u(x)|^2 dx \leq C_{n,p}\, \rho^{p-2} \int_{B_\rho} |\nabla u(x)|^p dx. \tag{2.4}$$

(b) *For* $p \geq 2$,

$$\int_{B_\rho} |u(x)|^{p-2}|\nabla u(x)|^2 dx \geq C_{n,p}\, \rho^{p-2} \int_{B_{\rho/2}} |\nabla u(x)|^p dx. \tag{2.5}$$

PROOF. We only prove the Lemma for $n \geq 3$, the special case $n = 2$ is similar.
(a) Set $\epsilon = \rho/4$ and $\delta = \rho/2$. Assume first that $u(x) \neq 0$ for all $x \in \overline{B(0,\rho)}$. Hence $|u|^p$ is C^2 for all $p > 1$ with

$$\Delta |u|^p = p(p-1)|u|^{p-2}|\nabla u|^2.$$

For $|x| \leq \epsilon$,

$$\begin{aligned} G_\delta(x) &= \frac{1}{n(n-2)\omega_n}\left[\frac{1}{|x|^{n-2}} - \frac{1}{\delta^{n-2}}\right] \\ &\geq \frac{1}{n(n-2)\omega_n}\left[\frac{4^{n-2}}{\rho^{n-2}} - \frac{2^{n-2}}{\rho^{n-2}}\right] = c_n \rho^{2-n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{B_\epsilon} |u(x)|^{p-2}|\nabla u(x)|^2 dx &\leq C_n \rho^{n-2} \int_{B_\delta} |u(x)|^{p-2}|\nabla u(x)|^2 G_\delta(x)\, dx \\ &= \frac{C_n}{p(p-1)}\rho^{n-2}\int_{B_\delta} \Delta |u(x)|^p G_\delta(x)\, dx. \end{aligned}$$

Hence by Lemma 2,

$$\int_{B_\epsilon} |u(x)|^{p-2}|\nabla u(x)|^2 dx \leq \rho^{n-2} C_{n,p}\left[\int_S |u(\delta\zeta)|^p d\sigma(\zeta) - |u(0)|^p\right]. \tag{2.6}$$

Set

$$I_1 = \int_S |u(\delta\zeta)|^p d\sigma(\zeta) - |u(0)|^p = \int_S \left(|u(\delta\zeta)|^2\right)^{p/2} d\sigma(\zeta) - \left(|u(0)|^2\right)^{p/2}.$$

Since $p \leq 2$,

$$\int_S (|u(\delta\zeta)|^2)^{p/2} d\sigma(\zeta) \leq \left(\int_S |u(\delta\zeta)|^2 d\sigma(\zeta)\right)^{p/2}.$$

Therefore

$$I_1 \leq \left(\int_S |u(\delta\zeta)|^2 d\sigma(\zeta)\right)^{p/2} - \left(|u(0)|^2\right)^{p/2}.$$

For $0 < \alpha \leq 1$ and $0 \leq b \leq a$ we have $a^\alpha - b^\alpha \leq (a-b)^\alpha$. Thus

$$I_1 \leq \left(\int_S |u(\delta\zeta)|^2 d\sigma(\zeta) - |u(0)|^2\right)^{p/2},$$

which by Lemma 2,

$$= \left(2\int_{B_\delta} |\nabla u(x)|^2 G_\delta(x)\right)^{p/2}$$
$$\leq C \sup_{x\in B_\delta} |\nabla u(x)|^p \left(\int_{B_\delta} G_\delta(x)\,dx\right)^{p/2}.$$

But

$$\int_{B_\delta} G_\delta(x)\,dx = c_n\rho^2.$$

Hence $I_1 \leq C\rho^p \sup_{x\in B_\delta} |\nabla u(x)|^p$. Since $|\nabla u|^p$ is subharmonic,

$$|\nabla u(x)|^p \leq \frac{c_n}{\delta^n}\int_{B(x,\delta)} |\nabla u(y)|^p dy \leq \frac{C}{\rho^n}\int_{B_\rho} |\nabla u(x)|^p dx$$

for all $x \in B_\delta$. Therefore,

$$I_1 \leq C\rho^{p-n}\int_{B_\rho} |\nabla u(x)|^p dx.$$

Combining this with inequality (2.6) gives

$$\int_{B_{\rho/4}} |u(x)|^{p-2}|\nabla u(x)|^2 dx \leq C_{n,p}\,\rho^{p-2}\int_{B_\rho} |\nabla u(y)|^p dy.$$

This proves inequality (2.4) whenever u is non-zero.

For arbitrary u, consider $u_\epsilon(x) = u(x) + i\epsilon$, $\epsilon > 0$. Then $u_\epsilon(x) \neq 0$ for all $x \in \overline{B(0,\rho)}$, and

$$\Delta|u_\epsilon|^p = p|u_\epsilon|^{p-4}|\nabla u|^2\left[(p-1)u^2 + \epsilon^2\right].$$

But $|\nabla u_\epsilon| = |\nabla u|$, and for $1 < p \leq 2$,

$$(p-1)|u_\epsilon|^2 \leq (p-1)u^2 + \epsilon^2 \leq |u_\epsilon|^2.$$

Hence

$$p(p-1)|u_\epsilon|^{p-2}|\nabla u_\epsilon|^2 \leq \Delta|u_\epsilon|^p \leq p|u_\epsilon|^{p-2}|\nabla u_\epsilon|^2.$$

The same proof as above now gives

$$\int_{B_{\rho/4}} |u_\epsilon(x)|^{p-2}|\nabla u_\epsilon(x)|^2 dx \leq C_{n,\rho}\rho^{n-2}\int_{B_\rho} |\nabla u(x)|^p dx.$$

Letting $\epsilon \to 0$ proves **(a)**.

(b) Suppose now that $p \geq 2$. Hence $|u|^p$ is C^2 for all such p. Since $(a-b)^\alpha \leq a^\alpha - b^\alpha$ whenever $0 \leq b \leq a$, $\alpha \geq 1$, we have

$$I_2 = \left(\int_S |u(\delta\zeta)|^2 d\sigma(\zeta)\right)^{p/2} - \left(|u(0)|^2\right)^{p/2}$$
$$\geq \left(\int_S |u(\delta\zeta)|^2 d\sigma(\zeta) - |u(0)|^2\right)^{p/2},$$

which by Lemma 2

$$= \left(2\int_{B_\delta} |\nabla u(x)|^2 G_\delta(x)\,dx\right)^{p/2}.$$

For $|x| \le \epsilon$ we have $G_\delta(x) \ge c_n \rho^{2-n}$. Therefore

$$I_2 \ge C\rho^{(2-n)p/2} \left(\int_{B_\epsilon} |\nabla u(x)|^2 dx \right)^{p/2}.$$

Since $|\nabla u(x)|^2$ is subharmonic,

$$\int_{B_\epsilon} |\nabla u(x)|^2 dx \ge c_n \epsilon^n |\nabla u(0)|^2 = C\rho^n |\nabla u(0)|^2.$$

Therefore $I_2 \ge C\rho^{(2-n)p/2} |\nabla u(0)|^p \rho^{pn/2} = C\rho^p\, |\nabla u(0)|^p$. By Lemma 2,

$$c_p \int_{B_\delta} |u(x)|^{p-2} |\nabla u(x)|^2 G_\delta(x) = \int_S \left(|u(\delta\zeta)|^2\right)^{p/2} d\sigma(\zeta) - \left(|u(0)|^2\right)^{p/2},$$

which since $p \ge 2$,

$$\ge \left(\int_S |u(\delta\zeta)|^2 d\sigma(\zeta) \right)^{p/2} - \left(|u(0)|^2\right)^{p/2}.$$

Combining the above gives

$$|\nabla u(0)|^p \le C\rho^{-p} \int_{B_\delta} |u(x)|^{p-2} |\nabla u(x)|^2 G_\delta(x)\, dx.$$

Since $G_\delta(x) \le C|x|^{2-n}$ we obtain

$$|\nabla u(0)|^p \le C\rho^{-p} \int_{B_\delta} |u(x)|^{p-2} |\nabla u(x)|^2 |x|^{2-n} dx. \tag{2.7}$$

For $w \in B_\delta$ set $v(x) = u(w+x)$. Then by (2.7)

$$|\nabla u(w)|^p \le C\rho^{-p} \int_{B_\delta} |u(w+x)|^{p-2} |\nabla u(w+x)|^2 |x|^{2-n} dx.$$

By the change of variables $y = w + x$, the above integral

$$= C\rho^{-p} \int_{B(w,\delta)} |u(y)|^{p-2} |\nabla u(y)|^2 |y-w|^{2-n} dy.$$

Therefore,

$$\int_{B_\delta} |\nabla u(w)|^p dw \le C\rho^{-p} \int_{B_\delta} \int_{B(w,\delta)} |u(y)|^{p-2} |\nabla u(y)|^2 |y-w|^{2-n} dy\, dw,$$

which by Fubini's theorem

$$\le C\rho^{-p} \int_{B_{2\delta}} |u(y)|^{p-2} |\nabla u(y)|^2 \int_{B(y,\delta)} |y-w|^{2-n} dw\, dy.$$

But

$$\int_{B(y,\delta)} |y-w|^{2-n} dw = \int_{B_\delta} |x|^{2-n} dx = c_n \rho^2.$$

Therefore

$$\int_{B_{\rho/2}} |\nabla u(w)|^p dw \le C\rho^{2-p} \int_{B_\rho} |u(y)|^{p-2} |\nabla u(y)|^2 dy.$$

This proves inequality (2.5). ∎

3. Proofs of the Main Results

Proof of Theorem 1. By Lemma 1

$$\int_\Omega \delta(x)^\gamma |h(x)|^{p-2}|\nabla h(x)|^2 dx \le C\int_\Omega \delta(w)^{\gamma-n}\left[\int_{B(w,\frac{1}{8}\delta(w))}|h(y)|^{p-2}|\nabla h(y)|^2 dy\right] dw.$$

Set $\rho = \frac{1}{2}\delta(w)$ and $u(x) = h(w+x)$. Then

$$\int_{B(w,\frac{1}{8}\delta(w))}|h(y)|^{p-2}|\nabla h(y)|^2 dy = \int_{B_{\rho/4}}|u(x)|^{p-2}|\nabla u(x)|^2 dx,$$

which by Lemma 3

$$\begin{aligned} &\le C_{n,p}\,\rho^{p-2}\int_{B_\rho}|\nabla u(x)|^p dx \\ &= C_{n,p}\delta(w)^{p-2}\int_{B(w)}|\nabla h(y)|^p dy. \end{aligned}$$

But by Lemma 1

$$\int_\Omega \delta(w)^{\gamma+p-2-n}\left[\int_{B(w)}|\nabla h(y)|^p dy\right] dw \le C\int_\Omega \delta(x)^{\gamma+p-2}|\nabla h(x)|^p dx.$$

This proves part (a). The proof of part (b) is similar. □

Before proving Theorem 2 we require some preliminary results about subharmonic functions. Let $\mathcal{S}^+(\Omega)$ denote the set of non-negative subharmonic functions on Ω that have a harmonic majorant on Ω. As in the Introduction, for $f \in \mathcal{S}^+(\Omega)$ we let H_f denote the least harmonic majorant of f on Ω. For convenience we will assume that $f \in C^2(\Omega)$. As in [**12**],[**13**] we have the following.

Lemma 4. *Let Ω be a domain in $\mathbb{R}^n$, $n \ge 2$, with Green function G, and let $f \in C^2(\Omega)$. Then $f \in \mathcal{S}^+(\Omega)$ if and only if there exists $t_o \in \Omega$ such that*

$$\int_\Omega G(t_o,x)\Delta f(x)\,dx < \infty. \tag{3.1}$$

If this is the case, then by the Riesz decomposition theorem

$$H_f(x) = f(x) + \int_\Omega G(x,y)\Delta f(y)\,dy \tag{3.2}$$

If the Green function G satisfies (1.3) and (1.4) then it is easily seen that (3.1) is equivalent to

$$\int_\Omega \delta(x)\Delta f(x)\,dx < \infty. \tag{3.3}$$

If the subharmonic function f is not C^2, then the quantity $\Delta f(x)\,dx$ may be replaced by $d\mu_f$, where μ_f is the Riesz measure of the subharmonic function f.

Lemma 5. *Let Ω be a domain in $\mathbb{R}^n$, $n \ge 2$, with Green function G satisfying (1.3) and (1.4), and let $t_o \in \Omega$ be arbitrary. Then there exists constants C_1 and*

C_2, depending only on t_o and Ω, such that for all $f \in \mathcal{S}^+(\Omega) \cap C^2(\Omega)$,

$$C_1 \left[f(t_o) + \int\limits_{\Omega} \delta(x)\Delta f(x)dx \right] \le H_f(t_o) \le C_2 \left[\int\limits_{B(t_o)} f(x)dx + \int\limits_{\Omega} \delta(x)\Delta f(x)dx \right].$$

PROOF. The left side of the previous inequality is an immediate consequence of (3.2) and inequality (1.3). The right side is obtained by integrating equation (3.2) over $B(t_o)$, and then proving that

$$\int_{B(t_o)} G(x,y)\,dx \le C\delta(y) \tag{3.4}$$

for all $y \in \Omega$, where C is a constant depending only on $\delta(t_o)$ and n. Inequality (3.4) follows from the fact that for $y \notin B(t_o)$, $x \to G(x,y)$ is harmonic on $B(t_o)$. For $y \in B(t_o)$, we use the fact that $G(x,y) \le c_n|x-y|^{2-n}$. For further details, the reader is referred to the proof of Proposition 4.3 of [**13**], where the result was proved for bounded domains in $\mathbb{R}^n$. The same proof works in general. □

Proof of Theorem 2. **(a)** Let h be a real-valued harmonic functions on Ω, $1 < p \le 2$, and let $t_o \in \Omega$ be fixed. Suppose first that $h(x) \ne 0$ for all $x \in \Omega$. Then $|h|^p$ is C^2 for all $p > 1$ with

$$\Delta |h|^p = p(p-1)|h|^{p-2}|\nabla h|^2.$$

By part (a) of Theorem 1 with $\gamma = 1$ we have

$$\int_{\Omega} \delta(x)\Delta|h(x)|^p dx \le C_p \int_{\Omega} \delta(x)^{p-1}|\nabla h(x)|^p dx. \tag{3.5}$$

Also, by Theorem 5.1 of [**13**]

$$\int_{B(t_o)} |h(x)|^p dx \le C \left[|h(t_o)|^p + \int_{B(t_o)} \delta_o(x)^p |\nabla h(x)|^p dx \right],$$

where $\delta_o(x)$ denotes the distance from $x \in B(t_o)$ to the boundary of $B(t_o)$. But for $x \in B(t_o)$ we have $\delta_o(x)^p \le \frac{1}{2}\delta(t_o)\delta(x)^{p-1}$. Thus

$$\int_{B(t_o)} |h(x)|^p dx \le C(t_o) \left[|h(t_o)|^p + \int_{B(t_o)} \delta(x)^{p-1} |\nabla h(x)|^p dx \right].$$

Combining this with inequality (3.5) and Lemma 5 gives

$$N_p^p(h) = H_{|h|^p}(t_o) \le C(t_o) \left[|h(t_o)|^p + \int_{\Omega} \delta(x)^{p-1} |\nabla h(x)|^p dx \right],$$

where $C(t_o)$ is a constant depending only on t_o and Ω. The proof of (a) for an arbitrary real-valued harmonic function h now follows by considering $h_\epsilon(x) = h(x) + i\epsilon$, $\epsilon > 0$, and then letting $\epsilon \to 0$. The proof of part **(b)** for $2 \le p < \infty$ is an immediate consequence of part (b) of Theorem 1 and Lemma 5. □

4. The Littlewood Paley Inequalties for Lipschitz domains

In this section we extend the Littlewood-Paley inequalities to bounded Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. As in [**1**], [**6**] a bounded domain Ω is a k-Lipschitz domain ($k > 0$) if Ω and its boundary $\partial\Omega$ are given locally by a Lipschitz function whose Lipschitz constant is less than or equal to k. For such domains one can find an exterior and interior cone at each boundary point with fixed aperture ψ. This aperture *psi* is related to the Lipschitz constant k by $\psi = \arctan(1/k) \in (0, \frac{1}{2}\pi)$. F-Y Maeda and N. Susuki proved ([**6**]) that for such domains Ω there exist constants α and β, $0 < \beta < 1 < \alpha$ such that the Green function G of Ω satisfies the following: for fixed $t_o \in \Omega$, there exist positive constants C_1, C_2, such that

$$C_1\delta(x)^\alpha \leq G(t_o, x) \qquad \text{for all } x \in \Omega, \text{ and} \tag{4.1}$$

$$G(t_0, x) \leq C_2\delta(x)^\beta \qquad \text{for all } x \in \Omega \setminus B(t_o, \tfrac{1}{2}\delta(t_o)), \tag{4.2}$$

The constants α and β are given by

$$\alpha = \alpha_n(\psi) \quad \text{and} \quad \beta = \alpha_n(\pi - \psi),$$

where α_n is a strictly decreasing continuous function on $(0, \pi)$ with $\alpha_n(\frac{1}{2}\pi) = 1$, $\alpha_n(\theta) \to \infty$ as $\theta \to 0$, and $\alpha_n(\theta) \to 0$ as $\theta \to \pi$ ($n \geq 3$). If $\partial\Omega$ is $C^{1,1}$, then $\alpha = \beta = 1$.

Using the same techniques as in the proof of Thoerem 2 we obtain the following version of the Littlewood-paley inequalities on bounded Lipschitz domains.

Theorem 4. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $\Omega \subsetneq \mathbb{R}^n$, with Lipschitz constant $k > 0$, and let G be the Green function of Ω satisfying (4.1) and (4.2) for fixed $t_o \in \Omega$. There exist constants C_1 and C_2, depending only on Ω, p, and k, such that for all real-valued harmonic functions h on Ω*
(a) *for $1 < p \leq 2$,*

$$N_p(h) \leq C_1 \left[|h(t_o)| + \left(\int_\Omega \delta(x)^{\beta+p-2} |\nabla h(x)|^p dx \right)^{1/p} \right].$$

(b) *If $h \in \mathcal{H}^p(\Omega)$, $2 \leq p < \infty$, then*

$$\left(|h(t_o)|^p + \int_\Omega \delta(x)^{\alpha+p-2} |\nabla h(x)|^p dx \right)^{1/p} \leq C_2 N_p(h).$$

5. The case $0 < p \leq 1$

In this section we prove a generalization of (1.1) for bounded domains in $\mathbb{R}^n$ with $C^{1,1}$ boundary for the case $0 < p \leq 1$. Suppose u is harmonic on Ω. When $0 < p < 1$, the function $|u(x)|^p$ is no longer subharmonic; in fact, if $u(x) \geq 0$, then u^p is superharmonic on Ω. As a consequence, the existence of a least harmonic majorant does not make sense. The fact that inequality (1.1) is valid for values of $p < 1$ on B should perhaps not be too surprising. For if $u(x) \geq 0$, then since u^p is superharmonic on B, we have

$$\int_S u^p(r\zeta)\, d\sigma(\zeta) \leq u^p(0)$$

for all r, $0 < r < 1$.

Suppose now that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{1,1}$ boundary. For each $t \in \partial\Omega$, let $\mathbf{n}_t$ denote the unit exterior normal at t, and for $r > 0$, set

$$\Omega_r = \{x \in \Omega : \delta(x) > r\}.$$

Since $\partial\Omega$ is $C^{1,1}$, there exists a $\rho_o > 0$ such that for each r, $0 < r < \rho_o$, $\partial\Omega_r$ is of class $C^{1,1}$ and the mapping $t \to t - r\mathbf{n}_t$ is a 1-1 transformation of $\partial\Omega$ onto $\partial\Omega_r$. (See [**2**], [**15**] for details.) We denote the element of surface measure on $\partial\Omega$ by dS.

For the proof of Theorem 3 we require the following generalization of an inequality of Fefferman and Stein.

LEMMA 6. *Let Ω be a proper open subset of $\mathbb{R}^n$ and let f be a non-negative subharmonic function on Ω. Then there exists a constant $C = C(n,p)$, depending only on n and p, such that*

$$f^p(x) \le \frac{C}{\delta(x)^n} \int_{B(x)} f^p(y)dy$$

for all $p > 0$.

Remark. Lemma 6 has previously been stated by Riihentaus in [**9**] and by Suzuki in [**14**]. For $p \ge 1$, the inequality follows immediately from the mean value property of subharmonic functions. For $0 < p < 1$ the result was proved in [**3**] for $|h|$, where h is harmonic on Ω. The inequality was also proved by Pavlović [**7**] for functions subharmonic with respect to the Laplace-Beltrami operator on the unit ball in $\mathbb{C}^n$. The same proof works in this setting, and thus is omitted.

Proof of Theorem 3. Let $0 < r < \rho_o$, and let h be harmonic on Ω. By the fundamental theorem of calculus, for fixed $\zeta \in \partial\Omega$,

$$f(\zeta - r\mathbf{n}_\zeta) - f(\zeta - \rho_o\mathbf{n}_\zeta) = \int_r^{\rho_o} \langle \nabla f(\zeta - t\mathbf{n}_\zeta), \mathbf{n}_\zeta \rangle \, dt.$$

Therefore

$$|f(\zeta - r\mathbf{n}_\zeta)| \le |f(\zeta - \rho_o\mathbf{n}_\zeta)| + \int_r^{\rho_o} |\nabla f(\zeta - t\mathbf{n}_\zeta)| \, dt. \tag{5.1}$$

For $k = 0, \ldots, N$, set $t_k = \rho_o 2^{-k}$, where N is the smallest integer such that $t_N < r$. Then

$$\begin{aligned}
\int_r^{\rho_o} |\nabla f(\zeta - t\mathbf{n}_\zeta)| \, dt &\le \sum_{k=1}^{N} \int_{t_k}^{t_{k-1}} |\nabla f(\zeta - t\mathbf{n}_\zeta)| \, dt \\
&\le \sum_{k=1}^{N} \sup_{t \in (t_k, t_{k-1}]} |\nabla f(\zeta - t\mathbf{n}_\zeta)|(t_{k-1} - t_k) \\
&= \sum_{k=1}^{N} t_k \sup_{t \in (t_k, t_{k-1}]} |\nabla f(\zeta - t\mathbf{n}_\zeta)|.
\end{aligned}$$

Hence, for $0 < p \le 1$,

$$|f(\zeta - r\mathbf{n}_\zeta)|^p \le |f(\zeta - \rho_o\mathbf{n}_\zeta)|^p + \sum_{k=1}^{N} t_k^p \sup_{t \in (t_k, t_{k-1}]} |\nabla f(\zeta - t\mathbf{n}_\zeta)|^p. \tag{5.2}$$

Since $|\nabla f|$ is subharmonic, by Lemma 6, for all $t \in (r, \rho_o]$,

$$|\nabla f(\zeta - t\mathbf{n}_\zeta)|^p \le \frac{C}{t^n} \int_{B(\zeta - t\mathbf{n}_\zeta)} |\nabla f(y)|^p \, dy \tag{5.3}$$

For each $k = 1, \ldots, N$ and $\zeta \in S$, set

$$B_k(\zeta) = B(\zeta - t_{k-1}\mathbf{n}_\zeta, \tfrac{7}{4}t_k).$$

We claim that $B(\zeta - t\mathbf{n}_\zeta) \subset B_k(\zeta)$ for all $t \in (t_k, t_{k-1}]$. Suppose $y \in B(\zeta - t\mathbf{n}_\zeta)$. Then

$$|y - (\zeta - t_{k-1}\mathbf{n}_\zeta)| \le |y - (\zeta - t\mathbf{n}_\zeta)| + |t_{k-1} - t| < \tfrac{1}{2}t + (t_{k-1} - t).$$

If $t \in (t_k, \frac{3}{2}t_k]$, then

$$|y - (\zeta - t_{k-1}\mathbf{n}_\zeta)| < \tfrac{3}{4}t_k + (t_{k-1} - t_k) = \tfrac{7}{4}t_k.$$

On the other hand, if $t \in (\frac{3}{2}t_k, t_{k-1}]$, then

$$|y - (\zeta - t_{k-1}\mathbf{n}_\zeta)| < \tfrac{1}{2}t_{k-1} + (t_{k-1} - \tfrac{3}{2}t_k) = \tfrac{3}{2}t_k,$$

which proves the claim. Also, for each $k = 1, \ldots, N$, we have

$$B_k(\zeta) \subset \Omega_{t_{k+2}} \setminus \Omega_{t_{k-2}}.$$

To see this, for $y \in B_k(\zeta)$ and $x \in \partial\Omega$,

$$\begin{aligned} |y - x| &\ge |(\zeta - t_{k-1}\mathbf{n}_\zeta) - x| - |y - (\zeta - t_{k-1}\mathbf{n}_\zeta)| \\ &> \delta(\zeta - t_{k-1}\mathbf{n}_\zeta) - \tfrac{7}{4}t_k = t_{k-1} - \tfrac{7}{4}t_k = t_{k+2}. \end{aligned}$$

Hence $\delta(y) > t_{k+2}$, i.e., $B_k(\zeta) \subset \Omega_{t_{k+2}}$. Also, for $y \in B_k(\zeta)$,

$$\delta(y) \le |y - \zeta| \le |y - (\zeta - t_{k-1}\mathbf{n}_\zeta)| + t_{k-1} < \tfrac{7}{4}t_k + 2t_k < t_{k-2}.$$

Hence $y \notin \Omega_{t_{k-2}}$. Thus $B_k(\zeta) \subset \Omega_{t_{k+2}} \setminus \Omega_{t_{k-2}}$. For convenience we set

$$A_k = \Omega_{t_{k+2}} \setminus \Omega_{t_{k-2}}.$$

From inequality (5.3) and the above we now obtain

$$\sup_{t \in (t_k, t_{k-1}]} |\nabla f(\zeta - t\mathbf{n}_\zeta)|^p \le \frac{C}{t_k^n} \int_{B_k(\zeta)} |\nabla f(y)|^p dy.$$

Thus by (5.2)

$$|f(\zeta - r\mathbf{n}_\zeta)|^p \le |f(\zeta - \rho_o\mathbf{n}_\zeta)|^p + C \sum_{k=1}^{N} t_k^{p-n} \int_{B_k(\zeta)} |\nabla f(y)|^p dy.$$

Integrating the above inequality over $\partial\Omega$ gives

$$\begin{aligned} \int_{\partial\Omega} |f(\zeta - r\mathbf{n}_\zeta)|^p dS(\zeta) \le &\int_{\partial\Omega} |f(\zeta - \rho_o\mathbf{n}_\zeta)|^p dS(\zeta) \\ &+ C \sum_{k=1}^{N} t_k^{p-n} \int_{\partial\Omega} \int_{B_k(\zeta)} |\nabla f(y)|^p dy \, dS(\zeta). \end{aligned}$$

For $y \in A_k$, let

$$\widetilde{B}_k(y) = \{\zeta \in \partial\Omega : y \in B_k(\zeta)\}.$$

Then $\chi_{B_k(\zeta)}(y) \le \chi_{\widetilde{B}_k(y)}(\zeta)$. Thus

$$\begin{aligned}\int_{\partial\Omega}\int_{B_k(\zeta)} |\nabla f(y)|^p dy\, dS(\zeta) &= \int_{\partial\Omega}\int_{A_k} \chi_{B_k(\zeta)}(y)|\nabla f(y)|^p dy\, dS(\zeta)\\ &\le \int_{\partial\Omega}\int_{A_k} \chi_{\widetilde{B}_k(y)}(\zeta)|\nabla f(y)|^p dy\, dS(\zeta),\end{aligned}$$

which by Fubini's theorem

$$\le \int_{A_k} |\widetilde{B}_k(y)||\nabla f(y)|^p dy,$$

where for $E \subset \partial\Omega$, $|E|$ denotes the surface measure of E.

Fix $y \in A_k$, and suppose $\zeta \in \widetilde{B}_k(y)$. Write $y = x - \delta(y)\mathbf{n}_x$, $x \in \partial\Omega$. Since $y \in B_k(\zeta)$,

$$\begin{aligned}|\zeta - x| &\le |(\zeta - t_{k-1}\mathbf{n}_\zeta) - y| + |t_{k-1}\mathbf{n}_\zeta - \delta(y)\mathbf{n}_x|\\ &< \tfrac{7}{4}t_k + |t_{k-1}\mathbf{n}_\zeta - \delta(y)\mathbf{n}_x|.\end{aligned}$$

Let $\theta(\zeta, x)$ denote the angle between the vectors $\mathbf{n}_\zeta$ and $\mathbf{n}_x$. Then

$$|t_{k-1}\mathbf{n}_\zeta - \delta(y)\mathbf{n}_x|^2 = (t_{k-1} - \delta(y))^2 + 4t_{k-1}\delta(y)\sin^2 \tfrac{1}{2}\theta(\zeta, x) \le Ct_k^2$$

for all $y \in A_k$. Therefore $\widetilde{B}_k(y) \subset B(x, Ct_k) \cap \partial\Omega$, and hence $|\widetilde{B}_k(y)| \le Ct_k^{n-1}$. Thus

$$\begin{aligned}t_k^{p-n}\int_{\partial\Omega}\int_{B_k(\zeta)} |\nabla f(y)|^p dy\, dS &\le Ct_k^{p-1}\int_{A_k} |\nabla f(y)|^p dy\\ &\le C\int_{A_k} \delta(y)^{p-1}|\nabla f(y)|^p dy.\end{aligned}$$

Finally, since

$$\sum_{k=1}^{N}\int_{A_k} \delta(y)^{p-1}|\nabla f(y)|^p \le C\int_\Omega \delta(y)^{p-1}|\nabla f(y)|^p dy,$$

we obtain

$$\int_{\partial\Omega} |f(\zeta - r\mathbf{n}_\zeta)|^p dS(\zeta) \le \int_{\partial\Omega} |f(\zeta - \rho_o\mathbf{n}_\zeta)|^p dS(\zeta) + C\int_\Omega \delta(y)^{p-1}|\nabla f(y)|^p dy$$

for all r, $0 < r \le \rho_o$.

To conclude the proof it remains to be shown that for $t_o \in \Omega$ fixed,

$$\int_{\partial\Omega} |f(\zeta - \rho_o\mathbf{n}_\zeta)|^p dS(\zeta) \le C\left[|f(t_o)|^p + \int_\Omega \delta(y)^{p-1}|\nabla f(y)|^p dy\right]. \tag{5.4}$$

Without loss of generality we assume that $t_o \in \Omega_{\rho_o}$. Then as in the proof of Theorem 5.1 [**13**], since $\overline{\Omega}_{\rho_o}$ is connected and $\partial\Omega_{\rho_o}$ is $C^{1,1}$, there exists a constant $C(\rho_o)$ such that for every $y \in \overline{\Omega}_{\rho_o}$ there exists a polygonal path $\gamma(t)$, $0 \le t \le 1$, in $\overline{\Omega}_{\rho_o}$ with $\gamma(0) = t_o$, $\gamma(1) = y$ and $\int_0^1 |\gamma'(t)|dt \le C(\rho_o)$. Thus

$$\begin{aligned}|f(y)| &\le |f(t_o)| + \int_0^1 |\nabla f(\gamma(t))||\gamma'(t)|\, dt\\ &\le |f(t_o)| + C(\rho_o)\sup\{|\nabla f(w)| : w \in \overline{\Omega}_{\rho_o}\}.\end{aligned}$$

Inequality (5.4) now follows by applying Lemma 6 to $|\nabla f(w)|$ on $B(w, \frac{1}{2}\rho_o)$, $w \in \overline{\Omega}_{\rho_o}$, and using the fact that $\frac{1}{2}\rho_o \le \delta(y) \le C$ for all $y \in \Omega_{\frac{1}{2}\rho}$. □

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