HARMONIC FUNCTION THEORY ON REAL HYPERBOLIC SPACE [1]

PRELIMINARY DRAFT

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Contents

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Introduction

The intend of these notes is to provide a detailed and comprehensive treatment of harmonic and subharmonic function theory on real hyperbolic space in $\mathbb{R}^n$. Although our primary emphasis will be in the setting of the unit ball B with hyperbolic metric ds given by

$$ds = \frac{2|dx|}{1-|x|^2}, \tag{0.1}$$

we will also consider the analogue of some of the results in the hyperbolic half–space $\mathbb{H}$. Undoubtedly some of the results are known, either in the setting of rank one non–compact symmetric spaces (e.g. [9]), or more generally, in Riemannian spaces (e.g. [4]). However, many of the results are new. Furthermore, our approach does not require any knowledge of Lie groups, and only limited knowledge of differential geometry. The goal is to make these notes accessible to a broad audience. Our development of the theory is analogous to the approach of W. Rudin ([17]) in his development of Möbius invariant harmonic function theory on the hermitian hyperbolic ball in $\mathbb{C}^n$.

In Chapter 1 we provide a brief review of Möbius transformation in $\mathbb{R}^n$. This is followed in Chapter 2 by a characterization of the group $\mathcal{M}(B)$ of Möbius self-maps of the unit ball B in $\mathbb{R}^n$. As in [17] we define a family $\{\varphi_a : a \in B\}$ of Möbius transformations of B satisfying $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a(\varphi_a(x)) = x$ for all $x \in B$. Furthermore, for every $\psi \in \mathcal{M}(B)$, it is proved that there exists $a \in B$ and an orthogonal transformation A such that $\psi = A\varphi_a$. When $n = 2$, the mappings φ_a

correspond to the usual analytic Möbius transformations of the unit disc $\mathbb{D}$ given by

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}. \tag{0.2}$$

Some of the properties of the mappings $\{\varphi_a\}$ and of functions in $\mathcal{M}(B)$ are developed in Section 2.1. In this chapter we also introduce the hyperbolic metric in B and also in the hyperbolic half–space $\mathbb{H}$.

In Chapter 3 we derive the Laplacian, gradient, and measure on B that are invariant under $\mathcal{M}(B)$. Even though the formula for the Laplacian can be derived from the hyperbolic metric, we will follow the approach of Rudin [17, Chapter 4]. For $f \in C^2(B)$ we define $\widetilde{\Delta} f$ by

$$\widetilde{\Delta} f(a) = \Delta(f \circ \varphi_a)(0).$$

The operator $\widetilde{\Delta}$ is shown to satisfy $\widetilde{\Delta}(f \circ \psi)(x) = (\widetilde{\Delta} f)(\psi(x))$ for all $\psi \in \mathcal{M}(B)$. Furthermore, an explicit computation gives

$$\widetilde{\Delta} f(x) = (1 - |x|^2)^2 \Delta f(x) + 2(n-2)(1 - |x|^2)\langle x, \nabla f(x)\rangle,$$

where ∇f is the euclidean gradient of the function f. In this chapter it is also proved that the Green's function for $\widetilde{\Delta}$ is given by $G(x, y) = g(|\varphi_x(y)|)$, where g is the radial function on B defined by

$$g(r) = \frac{1}{n} \int_r^1 \frac{(1 - s^2)^{n-2}}{s^{n-1}}\, ds.$$

In Theorem 3.4 we prove that for $\psi \in \mathcal{M}(B)$, the Jacobian J_ψ of the mapping ψ satisfies

$$|J_\psi(x)| = \frac{(1 - |\psi(x)|^2)^n}{(1 - |x|^2)^n}.$$

From this it now follows that the Möbius invariant measure τ on B is given by

$$d\tau(x) = (1 - |x|^2)^{-n} d\nu(x),$$

where ν is normalized volume measure on B. In Section 3.4 we include a brief discussion of the invariant Laplacian, gradient, and measure on the hyperbolic upper half–space $\mathbb{H}$.

A real–valued C^2 function f on B is defined to be either $\mathcal{M}$–harmonic or $\mathcal{M}$–subharmonic on B depending on whether $\widetilde{\Delta} f = 0$ or $\widetilde{\Delta} f \geq 0$. It is well know that a continuous function f is harmonic in the unit disc $\mathbb{D}$ if and only if for all r, $0 < r < 1$, and $w \in \mathbb{D}$,

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi_w(re^{it}))\, dt \tag{0.3}$$

where φ_w is the Möbius transformation of $\mathbb{D}$ given by (0.2). The above is called the *invariant mean–value property*. One of the first results proved in Chapter 4 is the following analogue of the invariant mean–value property: A real–valued C^2 function f is $\mathcal{M}$–subharmonic on B if and only if for all $a \in B$ and $0 < r < 1$,

$$f(a) \leq \int_S f(\varphi_a(rt)) d\sigma(t), \tag{0.4}$$

with equality if and only if f is $\mathcal{M}$–harmonic on B. In the above, S is the unit sphere in $\mathbb{R}^n$, σ is normalized surface measure on S, and φ_a is the Möbius transformation of B defined in Section 2.1. The integral in (0.4) is an average of f over the hyperbolic or non–euclidean sphere $\{\varphi_a(rt) : t \in S\}$ whose hyperbolic center is a. Inequality (0.4) is then used in Section 4.2 to extend the definition of $\mathcal{M}$–subharmonic to the class of upper semicontinuous functions on B.

The remainder of Chapter 4 is devoted to extending some of the standard results about subharmonic functions to $\mathcal{M}$–subharmonic functions on B. In Section 4.4 it is proved that every $\mathcal{M}$–subharmonic function on B is the limit of a decreasing sequence of C^∞ $\mathcal{M}$–subharmonic functions, and Section 4.5 provides a characterization of $\mathcal{M}$–subharmonic functions in terms of the "weak Laplacian". This is then used to prove the existence of the Riesz measure for an $\mathcal{M}$–subharmonic function.

The Poisson kernel P for $\widetilde{\Delta}$ is introduced in Chapter 5. In Section 5.1 we prove using Green's formula that for $(a, t) \in B \times S$,

$$P(a, t) = -\lim_{r \to 1} nr^{n-1}(1 - r^2)^{2-n} \langle \nabla G_a(rt), t \rangle,$$

where $G_a(rt) = G(a, rt)$ is the Green's function for $\widetilde{\Delta}$. This immediately gives

$$P(x, t) = \left(\frac{1 - |x|^2}{|x - t|^2} \right)^{n-1}.$$

The standard results for Poisson integrals of continuous functions are included in Section 5.2.

In Section 5.3 we derive the expansion of the Poisson kernel in terms of the zonal harmonics on B. One of the key results of this section is that if p_α is a spherical harmonic of degree α on S, then the Poisson integral $P[p_\alpha]$ of p_α is given by

$$P[p_\alpha](x) = |x|^\alpha S_{n,\alpha}(|x|) p_\alpha \left(\frac{x}{|x|} \right),$$

where $S_{n,\alpha}$ is given by a hypergeometric function. Interestingly, when n is even, $S_{n,\alpha}(r)$ is simply a polynomial in r of degree $n-2$. These results are then used to show how the Poisson integral $P[q]$ can be computed for any polynomial q on S. As an example, in $\mathbb{R}^4$, the $\mathcal{M}$–harmonic function with boundary values t_1^2 is given by $P[t_1^2](x) = \frac{1}{4} + (2 - |x|^2)(x_1^2 - \frac{1}{4}|x|^2)$. In contrast, the euclidean harmonic function h with boundary values t_1^2 is given by $h(x) = \frac{1}{4}(1 - |x|^2) + x_1^2$.

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1. Möbius Transformations

In this chapter we provide a brief review of Möbius transformations on n–dimensional euclidean space $\mathbb{R}^n$ $(n \geq 2)$. A good reference for these topics is the monograph by A. F. Beardon [3]. First however we begin with a review of notation that will be used throughout these notes.

1.1. Notation

For $x,\ y \in \mathbb{R}^n$ we let $\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$ denote the usual inner product on $\mathbb{R}^n$ and $|x| = \sqrt{\langle x, x\rangle}$ the length of the vector x. For $a \in \mathbb{R}^n$ and $r > 0$, the ball $B(a,r)$ and sphere $S(a,r)$ are given respectively by

$$\begin{aligned} B(a,r) &= \{x \in \mathbb{R}^n : |x-a| < r\}, \\ S(a,r) &= \{x \in \mathbb{R}^n : |x-a| = r\}. \end{aligned}$$

The unit ball and unit sphere with center at the origin will simply be denoted by B and S respectively. The *one–point compactification* of $\mathbb{R}^n$, denoted $\hat{\mathbb{R}}^n$, is obtained by appending the point ∞ to $\mathbb{R}^n$. A subset U of $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is open if it is an open subset of $\mathbb{R}^n$, or if U is the complement in $\hat{\mathbb{R}}^n$ of a compact subset C of $\mathbb{R}^n$. With this topology $\hat{\mathbb{R}}^n$ is compact.

For a subset D of $\mathbb{R}^n$, $\overline{D}$ denotes the closure of D, $\operatorname{Int}(D)$ the interior of D, ∂D the boundary of D, and $\widetilde{D}$ the complement of D in $\mathbb{R}^n$. Also if E and F are sets, $E \setminus F$ denotes the complement of F in E, i.e., $E \setminus F = E \cap \widetilde{F}$.

The study of functions of n–variables is simplified with the use of multi–index notation. For an ordered n–tuple $\alpha = (\alpha_1, ..., \alpha_n)$, where each α_j is a non-negative integer, the following notational conventions will be used throughout:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial X_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If Ω is an open subset of $\mathbb{R}^n$, we denote by $C^k(\Omega)$, $k = 0, 1, 2, ..$ the set of real–valued functions f on Ω for which $D^\alpha f$ exists and is continuous for all multi–indices α with $|\alpha| \leq k$. Thus $C^0(\Omega)$, or simply $C(\Omega)$, denotes the set of real–valued continuous functions on Ω, and $C^\infty(\Omega)$ the set of infinitely differentiable functions on Ω. Also, the set of functions $f \in C^k(\Omega)$ for which $D^\alpha f$, $|\alpha| \le k$, has a continuous extension to $\overline{\Omega}$ will be denoted by $C^k(\overline{\Omega})$. If $f : \Omega \mapsto \mathbb{R}$, then the *support* of f, denoted $\operatorname{supp} f$, is defined as

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The set of continuous functions on Ω with compact support will be denoted by $C_c(\Omega)$. The notations $C_c^k(\Omega)$ and $C_c^\infty(\Omega)$ have the obvious meanings.

A linear transformation $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be *orthogonal* if $|Ax| = |x|$ for all $x \in \mathbb{R}^n$. The set of orthogonal transformations of $\mathbb{R}^n$ will be denoted by $O(n)$. If A is represented by the $n \times n$ matrix $(a_{i,j})$, then A is orthogonal if and only if

$$\sum_{k=1}^n a_{i,k}a_{j,k} = \delta_{i,j} = \begin{cases} 1 & i = j, \\ 0, & i \neq j. \end{cases}$$

If $\psi(x) = (\psi_1(x), ..., \psi_n(x))$ is a C^1 mapping of an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, then the derivative $\psi'(x)$ is the $n \times n$ matrix given by

$$\psi'(x) = \left(\frac{\partial \psi_i}{\partial x_j}\right)_{i,j=1}^n,$$

and the *Jacobian* J_ψ of the transformation ψ is given by $J_\psi(x) = \det \psi'(x)$.

1.2. Inversion in Spheres and Planes[2]

Definition 1.1. *The inversion (or reflection) in the sphere $S(a, r)$ is the function $\phi(x)$ defined by*

$$\phi(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x - a). \tag{1.5}$$

For the unit sphere S, $\phi(x) = x/|x|^2$. The mapping $x \mapsto x/|x|^2$ is commonly denoted by x^*. Thus (1.5) can now be rewritten as

$$\phi(x) = a + r^2(x-a)^*.$$

The reflection $\phi(x)$ is not defined at $x = a$. Since $|\phi(x)| \to \infty$ as $x \to a$ we set $\phi(a) = \infty$. Also, since $\lim_{|x|\to\infty} |\phi(x) - a| = 0$, we set $\phi(\infty) = a$. Thus ϕ is defined on all of $\hat{\mathbb{R}}^n$, and it is easily shown that ϕ is continuous in the topology of $\hat{\mathbb{R}}^n$. A

[2]Although we will mainly be interested in the case $n \ge 2$, the formulas for inversions in spheres and planes are still meaningful when $n = 1$.

straightforward computation also shows that $\phi(\phi(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Thus ϕ is a one-to-one continuous map of $\hat{\mathbb{R}}^n$ onto $\hat{\mathbb{R}}^n$ satisfying $\phi(x) = x$ if and only $x \in S(a, r)$.

In addition to reflection in a sphere we also have reflection in a plane. For $a \in \mathbb{R}^n$, $a \neq 0$, and $t \in \mathbb{R}$, the plane $P(a, t)$ is defined by

$$P(a,t) = \{x \in \mathbb{R}^n : \langle x, a\rangle = t\}.$$

By convention ∞ belongs to every plane $P(a, t)$.

Definition 1.2. *The inversion (or reflection) in the plane $P(a, t)$ is the function $\psi(x)$ defined by*

$$\psi(x) = x + \lambda a,$$

where $\lambda \in \mathbb{R}$ is chosen so that $\frac{1}{2}(x + \psi(x)) \in P(a, t)$.

Solving for λ gives

$$\psi(x) = x - 2[\langle x, a\rangle - t]a^*, \quad x \in \mathbb{R}^n. \tag{1.6}$$

For the mapping ψ we have

$$|\psi(x)|^2 = |x|^2 + O(|x|),$$

and as a consequence $\lim\limits_{|x|\to\infty} |\psi(x)| = \infty$. Thus as above we define $\psi(\infty) = \infty$. With this definition the mapping ψ again satisfies $\psi(\psi(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Thus ψ is a one–to–one continuous map of $\hat{\mathbb{R}}^n$ onto itself with $\psi(x) = x$ if and only if $x \in P(a, t)$. It is well known that each inversion (in a sphere or a plane) is orientation–reversing and conformal (see [3, Theorem 3.1.6]).

1.3. Möbius Transformations

Definition 1.3. *A Möbius transformation of $\hat{\mathbb{R}}^n$ is a finite composition of inversions in spheres or planes.*

Clearly the composition of two Möbius transformations is again a Möbius transformation, as is the inverse of a Möbius transformation. The group of Möbius transformations on $\hat{\mathbb{R}}^n$ is called the *General Möbius group* and is denoted by GM $(\hat{\mathbb{R}}^n)$. Although not immediately obvious, both translation and magnification by a constant are Möbius transformations. The translation $x \mapsto x + a$, $a \in \mathbb{R}^n$, is the composition of inversion in the plane $\langle x, a\rangle = 0$ followed by inversion in the plane $\langle x, a\rangle = \frac{1}{2}|a|^2$. Likewise, the magnification or scalar multiplication $x \mapsto kx$, $k > 0$, is also a Möbius transformation in that is the inversion in S followed by the inversion in $S(0, \sqrt{k})$. Furthermore, every euclidean isometry of $\mathbb{R}^n$ is a composition of at most $n + 1$ reflections in planes ([3, Theorem 3.1.3]).

We conclude this section by showing that every every Möbius transformation maps a sphere or plane onto a sphere or plane. We will use the term "sphere" to denote either a sphere of the form $S(a,r)$ or a plane $P(a,t)$. Since every inversion ψ in a plane $P(a,t)$ can be written as

$$\psi(x) = x + \lambda a,$$

the mapping ψ clearly maps a "sphere" onto a "sphere". To show that an inversion ϕ in a sphere $S(a,r)$ preserves "spheres", it suffices to show that the mapping x^* preserves "spheres".

For any set $E \subset \mathbb{R}^n$, we let $E^* = \{x^* : x \in E\}$. A set $E \subset \mathbb{R}^n$ is a sphere or a plane if and only if

$$E = \{x \in \mathbb{R}^n : b|x|^2 - 2\langle x, a\rangle + c = 0\},$$

where b and c are real and $a \in \mathbb{R}^n$. By convention, ∞ satisfies this equation if and only if $b = 0$, that is, E is a plane. Now it is easily seen that E^* has the same form with the roles of b and c reversed. Finally, it is an easy exercise to show that for $a \in \mathbb{R}^n$ and $r > 0$,

$$S^*(a,r) = \begin{cases} S\left(\dfrac{a}{|a|^2 - r^2}, \dfrac{r}{||a|^2 - r^2|}\right) & \text{if } 0 \notin S(a,r), \\ P(a, \frac{1}{2}) & \text{if } 0 \in S(a,r). \end{cases} \tag{1.7}$$

We conclude this section with one more useful formula that will be required later. If ϕ is inversion in the sphere $S(a,r)$, then a straightforward computation gives

$$|\phi(y) - \phi(x)| = \frac{r^2|y - x|}{|x - a||y - a|}. \tag{1.8}$$

2. Möbius Self-Maps of the Unit Ball

In this chapter we will provide a characterization of the Möbius transformations of $\hat{\mathbb{R}}^n$ mapping the unit ball B onto B that is similar to the characterization of the Möbius mappings of the unit disc in $\mathbb{C}$ onto itself. The set of all Möbius transformations mapping B onto B will be denoted by $\mathcal{M}(B)$ or simply $\mathcal{M}$.

In the complex plane $\mathbb{C}$, every analytic Möbius transformation ψ mapping the unit disc $\mathbb{D}$ onto itself can be written as $\psi(z) = e^{i\theta}\varphi_w(z)$, where for $w \in \mathbb{D}$,

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}.$$

The mappings $\varphi_w(z)$ satisfy $\varphi_w(0) = w$, $\varphi_w(w) = 0$, and $\varphi_w(\varphi_w(z)) = z$ for all $z \in \mathbb{D}$. Furthermore, the mapping $\varphi_w(z)$ satisfies

$$1 - |\varphi_w(z)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2}.$$

2.1. Möbius Transformations of B

In this section we define an analogous family of Möbius transformations $\{\varphi_a : a \in B\}$ mapping B onto B having the property that every Möbius transformation ψ mapping B onto itself can be written as $\psi = A \circ \varphi_a$, where $a \in B$ and $A \in O(n)$. For $a \in B$, we first set

$$\psi_a(x) = a + (1 - |a|^2)(a - x)^*. \tag{2.1}$$

Since the mapping ψ_a is a composition of Möbius transformations, ψ_a is a Möbius transformation of $\hat{\mathbb{R}}^n$ mapping 0 to a^* and a to ∞. By a straightforward computation we have

$$|\psi_a(x)|^2 = \frac{|a - x|^2 + (1 - |a|^2)(1 - |x|^2)}{|a - x|^2},$$

and as a consequence

$$|\psi_a(x)|^2 - 1 = \frac{(1 - |a|^2)(1 - |x|^2)}{|x - a|^2}. \tag{2.2}$$

From the above it follows immediately that ψ_a maps B onto $\hat{\mathbb{R}}^n \setminus \overline{B}$.

We now define the mapping φ_a by

$$\varphi_a(x) = \psi_a(x)^* = \frac{\psi_a(x)}{|\psi_a(x)|^2}. \tag{2.3}$$

If we set[3]

$$\rho(x,a) = |x-a|^2 + (1-|a|^2)(1-|x|^2) = |a|^2|a^*-x|^2, \tag{2.4}$$

then the mapping φ_a can be expressed as

$$\varphi_a(x) = \frac{a|x-a|^2 + (1-|a|^2)(a-x)}{\rho(x,a)}. \tag{2.5}$$

As a consequence of (2.2)

$$1 - |\varphi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{\rho(x,a)}. \tag{2.6}$$

Thus φ_a is a Möbius transformation mapping B onto B with $\varphi_a(0) = a$ and $\varphi_a(a) = 0$. That φ_a maps B onto B follows immediately from the fact that ψ_a maps B onto $\hat{\mathbb{R}}^n \setminus \overline{B}$ and that x^* maps $\hat{\mathbb{R}}^n \setminus \overline{B}$ onto B. We will shortly prove that φ_a also satisfies $\varphi_a(\varphi_a(x)) = x$ for all $x \in B$. In the unit disk $\mathbb{D}$, $\rho(z,w) = |1-\overline{w}z|^2$ and the mappings $\varphi_w(z)$ as defined by (2.5) are precisely the functions $(w-z)\big/(1-\overline{w}z)$.

One of the advantages of the mappings φ_a is that the function $(a,x) \mapsto \varphi_a(x)$ is not only continuous on $\overline{B}\times\overline{B}$ but also differentiable in each of the variables. At this point we will include several computations involving derivatives of the mappings φ_a that will be required in the proof of Theorem 2.2 and also later in the sequel. Let $y_j(x)$ denote the jth coordinate of $y(x) = \varphi_a(x)$. Then by straightforward computations we have

$$\begin{aligned} &\frac{\partial y_j}{\partial x_i}(0) = -\delta_{i,j}(1-|a|^2), \qquad &&\frac{\partial y_j}{\partial x_i}(a) = \frac{-\delta_{i,j}}{(1-|a|^2)}, \\ &\frac{\partial^2 y_j}{\partial x_i^2}(0) = (1-|a|^2)[2a_j - 4a_i\delta_{i,j}]. \end{aligned} \tag{2.7}$$

Hence

$$\varphi_a'(0) = -(1-|a|^2)\, I \quad \text{and} \quad \varphi_a'(a) = -(1-|a|^2)^{-1} I,$$

where I is the $n \times n$ identity matrix.

Since the following theorem is well know we state it without proof. A proof may be found in [3, Theorem 3.4.1].

[3]In [3] the function $\sqrt{\rho(x,a)}$ is denoted by $[x,a]$.

Theorem 2.1. *Let ψ be a Möbius transformation of $\hat{\mathbb{R}}^n$ satisfying $\psi(0) = 0$ and $\psi(B) = B$. Then $\psi(x) = Ax$ for some orthogonal transformation A.*

We are now ready to state and proof the main result of this section.

Theorem 2.2. *For $a \in B$, let φ_a be defined by (2.5). Then*

(a) *φ_a is a one–to–one Möbius mapping of B onto B satisfying*

$$\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \textit{and} \quad \varphi_a(\varphi_a(x)) = x$$

for all $x \in B$.

(b) *If $\psi \in \mathcal{M}(B)$, then there exists an orthogonal transformation A and $a \in B$ such that $\psi(x) = A\varphi_a(x)$.*

Proof. (a) To prove (a) it only remains to be shown that $\varphi_a(\varphi_a(x)) = x$ for all $x \in B$. Set $\psi(x) = (\varphi_a \circ \varphi_a)(x)$. Then ψ is a Möbius transformation of $\hat{\mathbb{R}}^n$ mapping B onto B satisfying $\psi(0) = 0$. Thus $\psi(x) = Ax$ for some orthogonal transformation A. But then $A = \psi'(0)$. On the other-hand, by the chain rule and equations (2.7)

$$\psi'(0) = \varphi_a'(a)\varphi_a(0) = I.$$

Hence $A = I$ and thus $\varphi_a(\varphi_a(x)) = x$ for all $x \in B$.

(b) Let $\psi \in \mathcal{M}(B)$ and let $a = \psi^{-1}(0)$. Then $\psi \circ \varphi_a$ is a Möbius transformation of B that fixes the origin. Thus $\psi \circ \varphi_a(x) = Ax$ for some orthogonal transformation A. But then by (a) we have $\psi(x) = A\varphi_a(x)$. □

Prior to introducing the hyperbolic metric on B we prove an identity for mappings $\psi \in \mathcal{M}(B)$.

Theorem 2.3. *If $\psi \in \mathcal{M}(B)$, then for all $x, y \in B$,*

$$\frac{|\psi(x) - \psi(y)|^2}{(1 - |\psi(x)|^2)(1 - |\psi(y)|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

Proof. Although this identity could be proved using the mappings φ_a, it appears to be easier to use the mappings σ_a defined as follows: for $a \in B$, $a \neq 0$, let σ_a denote the inversion in the sphere $S(a^*, \sqrt{|a^*|^2 - 1})$, i.e.,

$$\sigma_a(x) = a^* + (|a^*|^2 - 1)(x - a^*)^*. \tag{2.8}$$

Then $\sigma_a(0) = a$, $\sigma_a(a) = 0$, and since σ_a is an inversion, $\sigma_a(\sigma_a(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Also, by identity (1.8),

$$|\sigma_a(x)|^2 = |\sigma_a(x) - \sigma_a(a)|^2 = \frac{(|a^*|^2 - 1)^2 |x - a|^2}{|x - a^*|^2 |a - a^*|^2},$$

which upon simplification gives

$$|\sigma_a(x)|^2 = \frac{|x-a|^2}{\rho(x,a)}.$$

Thus

$$1 - |\sigma_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{\rho(x,a)}. \tag{2.9}$$

Hence $\sigma_a \in \mathcal{M}(B)$.[4] Again by (1.8) we obtain

$$|\sigma_a(x) - \sigma_a(y)|^2 = \frac{(1-|a|^2)^2|x-y|^2}{\rho(x,a)\rho(y,a)}.$$

Combining this with (2.9) now gives

$$\frac{|\sigma_a(x)-\sigma_a(y)|^2}{(1-|\sigma_a(x)|^2)(1-|\sigma_a(y)|^2)} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}.$$

Finally, as in the proof of Theorem 2.2 (b), every $\psi \in \mathcal{M}(B)$ can be expressed as $\psi(x) = A\sigma_a(x)$ for some $A \in O(n)$ and $a \in B$. From this the result now follows. $\square$

As a consequence of the identity in Theorem 2.3, for $\psi \in \mathcal{M}(B)$,

$$\lim_{y\to x} \frac{|\psi(y)-\psi(x)|}{|y-x|} = \frac{1-|\psi(x)|^2}{1-|x|^2}. \tag{2.10}$$

This result will be required in proving the $\mathcal{M}$–invariance of the hyperbolic metric on B.

2.2. The Hyperbolic Metric on B

The *hyperbolic metric* ρ on B is given by

$$ds = \frac{2|dx|}{1-|x|^2}. \tag{2.11}$$

Thus if $\gamma : [0,1] \mapsto B$ is a C^1 curve in B, the *length* $L(\gamma)$ of γ is given by

$$L(\gamma) = \int_0^1 \frac{2|\gamma'(t)|\,dt}{1-|\gamma(t)|^2},$$

and for $a, b \in B$, the *hyperbolic distance* $d(a,b)$ between a and b is defined by

$$d(a,b) = \inf_\gamma L(\gamma),$$

where the infimum is taken over all C^1 curves $\gamma : [0,1] \mapsto B$ with $\gamma(0) = a$ and $\gamma(1) = b$. From this we immediately obtain that for $x \in B$,

$$d(0,x) = \log\left(\frac{1+|x|}{1-|x|}\right). \tag{2.12}$$

---

[4] Even though the mappings σ_a are easier to work with, they have the disadvantage that $\lim_{a\to 0} \sigma_a(x)$ does not exist.

Theorem 2.4. *For all $\psi \in \mathcal{M}(B)$ and $a, b \in B$, $d(\psi(a), \psi(b)) = d(a, b)$.*

Proof. To prove the theorem it suffices to prove that $L(\psi \circ \gamma) = L(\gamma)$ for all C^1 curves γ and $\psi \in \mathcal{M}(B)$. If we set $\sigma(t) = \psi(\gamma(t))$, then σ is a C^1 curve and

$$\begin{aligned} |\sigma'(t)| &= \lim_{h \to 0} \left| \frac{\sigma(t+h) - \sigma(t)}{h} \right| \\ &= \lim_{h \to 0} \frac{|\psi(\gamma(t+h)) - \psi(\gamma(t))|}{|h|}, \end{aligned}$$

which by (2.10)

$$= |\gamma'(t)| \left(\frac{1 - |\psi(\gamma(t))|^2}{1 - |\gamma(t)|^2} \right).$$

Thus

$$\frac{|\sigma'(t)|}{1 - |\sigma(t)|^2} = \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2}.$$

From this it now follows that $L(\sigma) = L(\gamma)$, thus proving the claim. $\square$

As a consequence of (2.12) and Theorem 2.4, for $a, b \in B$,

$$d(a, b) = d(0, \varphi_a(b)) = \log \left(\frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|} \right). \tag{2.13}$$

Some brief computations also give

$$\sinh^2 \tfrac{1}{2} d(a, b) = \frac{|a - b|^2}{(1 - |a|^2)(1 - |b|^2)},$$

or

$$d(a, b) = 2 \operatorname{Arcsinh} \left[\frac{|a - b|}{\sqrt{(1 - |a|^2)(1 - |b|^2)}} \right] = 2 \operatorname{Arctanh} \left[\frac{|a - b|}{\sqrt{\rho(a, b)}} \right], \tag{2.14}$$

where $\rho(a, b)$ is defined by (2.4).

For $0 < r < 1$ we will denote $B(0, r)$ and $S(0, r)$ by B_r and S_r respectively. As in [17, p. 29], for $a \in B$ and $0 < r < 1$, we let $E(a, r) = \varphi_a(B_r)$. Since φ_a is an involution,

$$E(a, r) = \{x \in B : |\varphi_a(x)| < r\} = \{x \in B : d(a, x) < \log \left(\frac{1 + r}{1 - r} \right) \}. \tag{2.15}$$

Thus $E(a, r)$ is a hyperbolic ball with hyperbolic center a and hyperbolic radius

$$\rho = \log \left(\frac{1 + r}{1 - r} \right) = 2 \operatorname{Arctanh} r.$$

However, $E(a, r)$ is also a euclidean ball whose center and radius are given in the following theorem.

Theorem 2.5. *For $a \in B$ and $0 < r < 1$, $E(a,r) = B(c_a, \rho_a)$ where*

$$c_a = \frac{(1-r^2)a}{(1-|a|^2r^2)} \quad \textit{and} \quad \rho_a = \frac{r(1-|a|^2)}{(1-|a|^2r^2)}.$$

Proof. To prove the result we first determine the image of S_r under the mapping φ_a. Let ψ_a be the mapping defined by (2.1). If $|a| \neq r$, then by (1.7)

$$\psi_a(S_r) = S\left(\frac{(1-r^2)a}{|a|^2-r^2}, \frac{(1-|a|^2)r}{||a|^2-r^2|}\right).$$

Since $\varphi_a = \psi_a^*$, using (1.7) again gives

$$\varphi_a(S_r) = S\left(\frac{(1-r^2)a}{(1-r^2|a|^2)}, \frac{(1-|a|^2)r}{(1-r^2|a|^2)}\right). \tag{2.16}$$

On the other-hand, if $|a| = r$, $\psi_a(S_r) = P(a, \frac{1}{2}(1+|a|^2))$. Taking the inversion of $\psi_a(S_r)$ gives

$$\varphi_a(S_r) = \{x : (1+|a|^2)|x|^2 - 2\langle a, x\rangle = 0\}.$$

This however is simply the equation of the sphere given in (2.16) with $r = |a|$. Since φ_a is continuous and $a \in B(c_a, \rho_a)$, $\varphi_a(B_r) \subset B(c_a, \rho_a)$. Finally, since φ_a is an involution, $\varphi_a(B_r) = B(c_a, \rho_a)$. $\square$

2.3. Hyperbolic Half–Space

In this final section we briefly consider hyperbolic half–space in $\mathbb{R}^n$.

Definition 2.6. *For $n \geq 2$, the upper half–space $\mathbb{H}$ or $\mathbb{H}_n$ in $\mathbb{R}^n$ is defined by*

$$\mathbb{H} = \{x \in \mathbb{R}^n : x_n > 0\}.$$

For each $x \in \mathbb{R}^{n-1}$, let $\tilde{x} \in \mathbb{R}^n$ be defined by

$$\tilde{x} = (x, 0) = (x_1, ..., x_{n-1}, 0).$$

For each inversion ϕ on $\hat{\mathbb{R}}^{n-1}$, we define an inversion $\tilde{\phi}$ acting on $\hat{\mathbb{R}}^n$ as follows. If ϕ is an inversion in $S(a,r)$, then $\tilde{\phi}$ is the inversion in $S(\tilde{a}, r)$: if ϕ is an inversion in $P(a,t)$, then $\tilde{\phi}$ is the inversion in $P(\tilde{a}, t)$. If $x \in \mathbb{R}^{n-1}$, then

$$\tilde{\phi}(\tilde{x}) = \tilde{\phi}(x, 0) = (\phi(x), 0) = \widetilde{\phi(x)}.$$

The function $\tilde{\phi}$ is called the *Poincaré extension* of ϕ.

Suppose ϕ is an inversion in $S(a,r)$, $a \in \mathbb{R}^{n-1}$. Then for $x \in \mathbb{R}^n$, $\tilde{\phi}(x) = \tilde{a} + r^2(x-\tilde{a})^*$. If $[\tilde{\phi}(x)]_j$ denotes the jth coordinate function of $\tilde{\phi}(x)$, then

$$[\tilde{\phi}(x)]_n = \frac{r^2 x_n}{|x-\tilde{a}|^2}.$$

By (1.8) and the above

$$\frac{|\tilde{\phi}(x)-\tilde{\phi}(y)|^2}{[\tilde{\phi}(x)]_n[\tilde{\phi}(y)]_n} = \frac{|x-y|^2}{x_n y_n}.$$

As a consequence the mapping $\tilde{\phi}$ leaves

$$\frac{|x-y|^2}{x_n y_n} \tag{2.17}$$

invariant. If ϕ is reflection in the plane $P(a,t)$, then $\tilde{\phi}$ is a euclidean isometry of $\mathbb{R}^n$ with $[\tilde{\phi}(x)]_n = x_n$. Thus (2.17) is also invariant under $\tilde{\phi}$. Furthermore, in both cases we have $\tilde{\phi}(\mathbb{H}) = \mathbb{H}$.

If ϕ is any Möbius transformation acting on $\hat{\mathbb{R}}^{n-1}$, i.e., $\phi = \phi_1 \circ \cdots \circ \phi_m$, where each ϕ_j is an inversion in $\mathbb{R}^{n-1}$, then $\tilde{\phi} = \tilde{\phi}_1 \circ \cdots \circ \tilde{\phi}_m$ is an extension of ϕ to a Möbius transformation of $\hat{\mathbb{R}}^n$ which preserves $\mathbb{H}$. By Theorem 3.2.4 of [3] this extension is unique. Also, since each $\tilde{\phi}_j$ leaves (2.17) invariant, so does the mapping $\tilde{\phi}$. As a consequence the Poincaré extension $\tilde{\phi}$ of any $\phi \in GM(\hat{\mathbb{R}}^{n-1})$ is an isometry of the half–space $\mathbb{H}_n$ when endowed with the Riemannian metric d given by

$$ds = \frac{|dx|}{x_n}.$$

This metric is invariant under $\tilde{\phi}$ for each $\phi \in GM(\hat{\mathbb{R}}^{n-1})$. For this metric, we have for $x, y \in \mathbb{H}_n$,

$$\sinh^2 \tfrac{1}{2} d(x,y) = \frac{|x-y|^2}{4 x_n y_n},$$

or

$$d(x,y) = 2\,\mathrm{Arcsinh}\left[\frac{|x-y|}{2\sqrt{x_n y_n}}\right].$$

We conclude this section by considering the Möbius transformation Φ that maps $\mathbb{H}$ onto B. Set $e_n = (0,...,0,1)$ and let Φ denote the inversion in $S(-e_n, \sqrt{2})$, i.e.,

$$\Phi(x) = -e_n + \frac{2(x+e_n)}{|x+e_n|^2}. \tag{2.18}$$

Then

$$\begin{aligned}|\Phi(x)|^2 &= 1 + \frac{4}{|x+e_n|^2} - \frac{4\langle e_n, x+e_n\rangle}{|x+e_n|^2}\\ &= 1 - \frac{4x_n}{|x+e_n|^2},\end{aligned}$$

or

$$1 - |\Phi(x)|^2 = \frac{4x_n}{|x+e_n|^2}.$$

Since Φ is an inversion, Φ is a one–to–one map of $\hat{\mathbb{R}}^n$ onto $\hat{\mathbb{R}}^n$ satisfying $\Phi(\Phi(x)) = x$ for all $x \in \mathbb{R}^n$. Also, since $|\Phi(x)| = 1$ when $x_n = 0$, Φ maps $\mathbb{H}$ onto B, B onto $\mathbb{H}$, and $\partial\mathbb{H}$ (in $\hat{\mathbb{R}}^n$) onto S.

Finally, since Φ is the inversion in $S(-e_n, \sqrt{2})$, by Identity (1.8)

$$|\Phi(y) - \Phi(x)| = \frac{2|y-x|}{|y+e_n||x+e_n|},$$

and thus

$$\begin{aligned}\lim_{y\to x} \frac{|\Phi(y)-\Phi(x)|}{|y-x|} &= \frac{2}{|x+e_n|^2}\\ &= \frac{1-|\Phi(x)|^2}{2x_n}.\end{aligned} \tag{2.19}$$

We conclude this section by computing $J_\Phi(x)$. For $\phi(x) = x^*$, we have

$$\phi'(x) = |x|^{-4}[|x|^2 I - 2Q(x)],$$

where $Q(x)$ is the $n \times n$ symmetric matrix $(x_i x_j)_{i,j=1}^n$. Since the characteristic polynomial of $2Q(x)$ is $s^{n-1}(s - 2|x|^2)$, taking $s = |x|^2$ gives

$$\det \phi'(x) = -|x|^{-2n}.$$

But $\Phi(x) = -e_n + 2\phi(x+e_n)$. Thus $\Phi'(x) = 2\phi'(x+e_n)$ and hence

$$J_\Phi(x) = \det \Phi'(x) = -\frac{2^n}{|x+e_n|^{2n}}. \tag{2.20}$$

3. The Invariant Laplacian, Gradient, and Measure

In order to study harmonic function theory on the hyperbolic ball B, we first need to determine the Laplacian $\widetilde{\Delta}$, gradient $\widetilde{\nabla}$, and measure τ on B that are invariant under the group $\mathcal{M}(B)$ of Möbius transformations of B. Although these are well known in the setting of rank one non–compact symmetric spaces, we follow the approach of Rudin [17, Chapter 4] to determine $\widetilde{\Delta}$ and $\widetilde{\nabla}$.

3.1. The Invariant Laplacian and Gradient

Definition 3.1. *Suppose Ω is an open subset of B, $f \in C^2(\Omega)$ and $a \in \Omega$. We define*

$$(\widetilde{\Delta} f)(a) = \Delta(f \circ \varphi_a)(0),$$

where φ_a is the involution defined by (2.3) and $\Delta = \sum \frac{\partial}{\partial x_j^2}$ is the usual Laplacian on $\mathbb{R}^n$. The operator $\widetilde{\Delta}$ is called the invariant Laplacian or Laplace-Beltrami operator on B. Also, for $f \in C^1(\Omega)$, we define the invariant gradient $\widetilde{\nabla}$ by

$$(\widetilde{\nabla} f)(a) = -\nabla(f \circ \varphi_a)(0),$$

where $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$ is the usual gradient.[5]

Suppose f is a C^1 (or C^2) function on B and $y = \psi(x)$ is a C^1 (or C^2) mapping of B into B. Then if $g = f \circ \psi$, by the chain rule

$$\nabla g(x) = \psi'(x) \nabla f(\psi(x)),$$

and

$$\Delta g(x) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial y_i \partial y_j} \langle \nabla y_i, \nabla y_j \rangle + \sum_{j=1}^{n} \frac{\partial f}{\partial y_j} \Delta y_j.$$

Hence if $y = \varphi_a(x)$, from equations (2.7) it now follows that

$$\widetilde{\nabla} f(a) = (1 - |a|^2) \nabla f(a) \tag{3.1}$$

[5]The choice of the minus sign in the definition of $\widetilde{\nabla}$ will assure that $\widetilde{\nabla} f$ is in the same direction as ∇f.

and

$$\widetilde{\Delta} f(a) = (1 - |a|^2)^2 \Delta f(a) + 2(n-2)(1 - |a|^2)\langle a, \nabla f(a)\rangle. \text{ [6]} \tag{3.2}$$

If f is a radial function, i.e. $f(x) = g(|x|)$, then with $r = |x|$,

$$\widetilde{\nabla} f(x) = (1 - r^2) g'(r) \frac{x}{r} \tag{3.3}$$

and thus $|\widetilde{\nabla} f(x)| = (1 - r^2) g'(r)$. Also,

$$\widetilde{\Delta} f(x) = (1 - r^2) \left[(1 - r^2) g''(r) + \frac{g'(r)}{r} \{(n-1)(1 - r^2) + 2(n-2) r^2\} \right]. \tag{3.4}$$

The Möbius invariant Laplacian on the hermitian hyperbolic ball B in $\mathbb{C}^n$ (see [17, Chapter 4]) is given by

$$\widetilde{\Delta} f(z) = 4(1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{i,j} - z_i \overline{z}_j) \frac{\partial^2 f(z)}{\partial z_i \partial \overline{z}_j}.$$

In contrast to the Laplacian on real-hyperbolic space, this operator has no linear terms. The following theorem justifies the term "invariant" in reference to the operator $\widetilde{\Delta}$ and the gradient $\widetilde{\nabla}$.

Theorem 3.2. *For $f \in C^2(\Omega)$ and $\psi \in \mathcal{M}(B)$,*

$$\widetilde{\Delta}(f \circ \psi) = (\widetilde{\Delta} f) \circ \psi \quad \text{and} \quad |\widetilde{\nabla}(f \circ \psi)| = |(\widetilde{\nabla} f) \circ \psi|.$$

Proof. As in [17, Theorem 4.12], pick $b \in \psi^{-1}(\Omega)$ and put $a = \psi(b)$. Then $\varphi_a \circ \psi \circ \varphi_b$ is a Möbius transformation of B fixing 0. Thus $\psi \circ \varphi_b = \varphi_a \circ A$ for some orthogonal transformation A. Hence

$$\widetilde{\Delta}(f \circ \psi)(b) = \Delta(f \circ \psi \circ \varphi_b)(0) = \Delta(f \circ \varphi_a \circ a)(0).$$

[6] If the hyperbolic metric (2.11) is expressed in standard Riemannian notation as

$$ds^2 = \sum_{i,j} g_{i,j} dx_i dx_j,$$

where $g_{i,j} = 4\delta_{i,j}(1 - |x|^2)^{-2}$, then the Laplace-Beltrami operator L on B is defined by

$$L(f) = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_i (g^{i,j} \sqrt{g} \partial_j f),$$

where $g = \det(g_{i,j})$, and $(g^{i,j})$ is the inverse matrix of $(g_{i,j})$. A brief computation shows that except for a factor of $\frac{1}{4}$, this agrees with the operator $\widetilde{\Delta}$ given by (3.2).

But for any orthogonal transformation A, by the computations following Definition 3.1,

$$\Delta(g \circ A)(0) = (\Delta g)(0).$$

Therefore, $\Delta(f \circ \varphi_a \circ A)(0) = \Delta(f \circ \varphi_a)(0) = \widetilde{\Delta} f(a)$, that is, $\widetilde{\Delta}(f \circ \psi)(b) = (\widetilde{\Delta} f)(\psi(b))$. An analogous argument proves that $|\widetilde{\nabla}(f \circ \psi)| = |(\widetilde{\nabla} f) \circ \psi|$. $\square$

3.2. The Fundamental Solution of $\widetilde{\Delta}$

Suppose g is a radial solution of $\widetilde{\Delta} g = 0$. If we let $v(r) = g'(r)$, then by (3.4) the function v must satisfy

$$(1-r^2)v'(r) + \{(n-1)(1-r^2) + 2(n-2)r^2\}\frac{v(r)}{r} = 0,$$

or

$$\frac{v'(r)}{v(r)} = -(n-1)\frac{1}{r} - (n-2)\frac{2r}{1-r^2}.$$

Solving this differential equation for $v(r)$ gives

$$v(r) = c\frac{(1-r^2)^{n-2}}{r^{n-1}}.$$

for some constant c. Thus

$$g(r) = \frac{1}{n}\int_r^1 \frac{(1-s^2)^{n-2}}{s^{n-1}}\,ds \tag{3.5}$$

is a radial solution for $\widetilde{\Delta}$. The choice of the constant $\frac{1}{n}$ will become apparent in Theorem 4.2. When $n = 2$ this gives the usual solution

$$g(r) = \tfrac{1}{2}\log\tfrac{1}{r}.$$

When $n > 2$, by estimating the integral in (3.5) we obtain the existence of constants c_1 and c_2, independent of x, such that

$$c_1\frac{(1-|x|^2)^{n-1}}{|x|^{n-2}} \le g(|x|) \le c_2\frac{(1-|x|^2)^{n-1}}{|x|^{n-2}} \tag{3.6}$$

for all $x \in B$, $x \neq 0$. When $n = 2$, the fundamental solution $g(|x|) = -\frac{1}{2}\log|x|$ satisfies

$$\tfrac{1}{2}(1-|x|) \le g(x) \le \frac{(1-|x|)}{2|x|}. \tag{3.7}$$

Definition 3.3. *For $x, y \in B$, $x \neq y$, the Green's function $G(x,y)$ for $\widetilde{\Delta}$ is defined by*

$$G(x,y) = g(|\varphi_y(x)|) = \frac{1}{n}\int_{|\varphi_y(x)|}^{1} \frac{(1-s^2)^{n-2}}{s^{n-1}}\,ds.$$

Since $|\varphi_y(x)| = |\varphi_x(y)|$, we have $G(x,y) = G(y,x)$ for all $x, y \in B$, $x \neq y$. Also, if $y \in B$ is fixed, the function $x \mapsto G(x,y)$ satisfies $\widetilde{\Delta}_x G(x,y) = 0$ on $B \setminus \{y\}$. Furthermore, by (3.6), for $n > 2$,

$$G(x,y) \approx^{7} \frac{(1-|\varphi_y(x)|^2)^{n-1}}{|\varphi_y(x)|^{n-2}}$$

for all $x, y \in B$, $x \neq y$. Thus by (2.2), for $n > 2$ we have

$$G(x,y) \approx \frac{(1-|x|^2)^{n-1}(1-|y|^2)^{n-1}}{|x-y|^{n-2}\{|x-y|^2 + (1-|x|^2)(1-|y|^2)\}^{n/2}}. \tag{3.8}$$

3.3. The Invariant Measure on B

Our next step is to determine the Möbius invariant measure τ on B. First however we introduce some notation and formulas concerning integration on B. We denote by ν Lebesgue measure in $\mathbb{R}^n$ normalized so that $\nu(B) = 1$. Also, we denote by σ surface measure on S again normalized such that $\sigma(S) = 1$. Then by integration in polar coordinates we have

$$\int_B f(x)\,d\nu(x) = n\int_0^1 r^{n-1}\int_S f(r\zeta)\,d\sigma(\zeta)dr. \tag{3.9}$$

The measure σ is invariant under $O(n)$, that is,

$$\int_S f(A\zeta)d\sigma(\zeta) = \int_S f(\zeta)d\sigma(\zeta) \tag{3.10}$$

for all $A \in O(n)$ and $f \in L^1(S)$. Furthermore, if K is any compact subgroup of $O(n)$,

$$\int_S f(\zeta)d\sigma(\zeta) = \int_S\int_K f(k\zeta)\,dk d\sigma(\zeta), \tag{3.11}$$

[7] If f and g are real-valued functions on a set S, we use the notation $f(x) \approx g(x)$ to mean that there exist constants c_1 and c_2, independent of x, such that

$$c_1 f(x) \leq g(x) \leq c_2 f(x)$$

for all $x \in S$.

where dk denotes Haar measure on K.

To determine the invariant measure τ on B we assume that $d\tau(x) = \rho(x)d\nu(x)$, where ρ is a radial function on B. Then for $f \in L^1(B,\tau)$ and the mapping $\varphi_a \in \mathcal{M}(B)$ we have

$$\int_B (f \circ \varphi_a) d\tau = \int_B f(\varphi_a(x))\rho(x)d\nu(x),$$

which by the change of variables formula for $\mathbb{R}^n$

$$= \int_B f(x)\rho(\varphi_a(x))|J_{\varphi_a}(x)|d\nu(x),$$

where J_{φ_a} is the Jacobian of the mapping φ_a. Thus in order that $\int (f \circ \varphi_a)d\tau = \int f d\tau$ for all $f \in L^1(B,\tau)$ and φ_a, we must have $\rho(\varphi_a(x))|J_{\varphi_a}(x)| = \rho(x)$. In particular with $x = 0$, by (2.7), $\rho(a) = (1-|a|^2)^{-n}\rho(0)$. Hence we define the measure τ on B by

$$d\tau(x) = \frac{d\nu(x)}{(1-|x|^2)^n}. \tag{3.12}$$

In the following theorem we prove that τ is the *Möbius invariant measure* on B.

Theorem 3.4. **(a)** *If $\psi \in \mathcal{M}(B)$, then the Jacobian J_ψ of ψ satisfies*

$$|J_\psi(x)| = \frac{(1-|\psi(x)|^2)^n}{(1-|x|^2)^n}$$

for all $x \in B$.

(b) *The measure τ defined by (3.12) satisfies*

$$\int_B f\, d\tau = \int_B (f \circ \psi)\, d\tau$$

for every $f \in L^1(B,\tau)$ and $\psi \in \mathcal{M}(B)$.

Proof. (a) For $\psi \in \mathcal{M}(B)$ and $a \in B$, let $b = \psi(a)$. Then $\varphi_b \circ \psi \circ \varphi_a \in \mathcal{M}(B)$ and fixes 0. Thus $\psi(x) = \varphi_b \circ A \circ \varphi_a(x)$ for some $A \in O(n)$. But then $\psi'(x) = \varphi_b'(A\varphi_a(x))A\varphi_a'(x)$, or

$$\psi'(a) = \varphi_b'(0)A\varphi_a'(a).$$

Hence by (2.7)

$$\psi'(a) = \frac{(1-|b|^2)}{(1-|a|^2)}A.$$

Since A is orthogonal, $|\det A| = 1$. Therefore

$$|J_\psi(a)| = |\det \psi'(a)| = \frac{(1-|\psi(a)|^2)^n}{(1-|a|^2)^n}.$$

For (b), if $f \in L^1(B,\tau)$ and $\psi \in \mathcal{M}(B)$, then

$$\int_B f\,d\tau = \int_B f(w)(1-|w|^2)^{-n} d\nu(w),$$

which by the change of variables formula

$$\begin{aligned} &= \int_B f(\psi(x))(1-|\psi(x)|^2)^{-n}|J_\psi(x)|d\nu(x) \\ &= \int_B f(\psi(x))(1-|x|^2)^{-n}d\nu(x) = \int_B (f\circ\psi)\,d\tau. \quad \square \end{aligned}$$

For future reference we estimate $\tau(E(a,r))$, $0<r<1$. Since $E(a,r) = \varphi_a(B_r)$,

$$\tau(E(a,r)) = \tau(B_r) = n\int_0^r \frac{\rho^{n-1}}{(1-\rho^2)^n} d\rho.$$

For $n \geq 2$, set

$$c_n(r) = \frac{r^n}{(1-r^2)^{n-1}} \tag{3.13}$$

Then $c_2(r) = \tau(E(a,r))$, and for $n>2$, by L'Hospital's rule,

$$\begin{aligned} \lim_{r\to1}\frac{\tau(B_r)}{c_n(r)} &= \lim_{r\to1}\frac{n}{n(1-r^2)+2(n-1)r^2} \\ &= \frac{n}{2(n-1)}, \qquad (3.14)\\ \lim_{r\to0}\frac{\tau(B_r)}{c_n(r)} &= 1. \end{aligned}$$

Thus for all $n\geq2$,

$$\tau(B_r) = \tau(E(a,r)) \approx \frac{r^n}{(1-r^2)^{n-1}}, \tag{3.15}$$

with equality when $n=2$.

3.4. The Invariant Laplacian, Gradient, and Measure in Hyperbolic Half-Space

Details to be completed!!!!!!

For $f \in C^2(\mathbb{H})$, the invariant Laplacian L and invariant gradient $\widetilde{\nabla}$ are given by

$$Lf(y) = y_n^2 \Delta f(y) - (n-2) y_n \frac{\partial f}{\partial y_n}$$

and

$$\widetilde{\nabla} f(y) = y_n \nabla f.$$

The Möbius invariant measure τ on $\mathbb{H}$ is given by

$$d\tau(y) = c_n y_n^{-n} dy$$

for an appropriate constant c_n.

4. Möbius Invariant Harmonic and Subharmonic Functions

In this chapter we consider the class of functions on B that are harmonic or subharmonic with respect to the operator $\widetilde{\Delta}$. We begin with the following definition.

Definition 4.1. *Let Ω be an open subset of B. A function $f \in C^2(\Omega)$ is said to be $\mathcal{M}$–harmonic (or invariant harmonic) on Ω if $\widetilde{\Delta} f(x) = 0$ for all $x \in \Omega$. Also, f is said to be $\mathcal{M}$–subharmonic on Ω if $\widetilde{\Delta} f(x) \geq 0$ for all $x \in \Omega$.*

Clearly if f is $\mathcal{M}$–harmonic of $\mathcal{M}$–subharmonic on B, then by the $\mathcal{M}$–invariance of $\widetilde{\Delta}$ so is $f \circ \psi$ for all $\psi \in \mathcal{M}(B)$. In Section 4.2 we extend the definition of $\mathcal{M}$–subharmonic function to the class of upper semicontinuous functions.

4.1. The Invariant Mean–Value Property

Recall that a function f on B is radial if $f(x) = g(|x|)$ for some function g on $[0, 1)$. This is equivalent to $f(Ax) = f(x)$ for all $A \in O(n)$. For a continuous function f on B we define the *radialization* $f^\sharp$ of f by

$$f^\sharp(x) = \int_{O(n)} f(Ax)\, dA = \int_S f(|x|\zeta) d\sigma(\zeta), \quad x \in B, \tag{4.1}$$

where dA is the Haar measure on $O(n)$. Clearly $f^\sharp$ is radial on B, and if $f \in C^2(B)$,

$$\widetilde{\Delta} f^\sharp(x) = \int_{O(n)} (\widetilde{\Delta} f)(Ax) dA.$$

Prior to proving the mean-value property for $\mathcal{M}$–harmonic and $\mathcal{M}$–subharmonic functions we prove the following analogue of Lemma 2.5 of [16].

Theorem 4.2. *If $f \in C^2(B)$, then*

(a) $\displaystyle \frac{d}{dr} \int_S f(r\zeta) d\sigma(\zeta) = \tfrac{1}{n} r^{1-n} (1 - r^2)^{n-2} \int_{B_r} \widetilde{\Delta} f(x) d\tau(x),$

and

(b) $\displaystyle f(0) = \int_S f(r\zeta) d\sigma(\zeta) - \int_{B_r} g(|x|, r) \widetilde{\Delta} f(x)\, d\tau(x),$

where

$$g(|x|,r)=\frac{1}{n}\int_{|x|}^{r}\frac{(1-s^2)^{n-2}}{s^{n-1}}\,ds \tag{4.2}$$

Proof. Since $(\widetilde{\Delta}f)^\sharp=\widetilde{\Delta}(f^\sharp)$, to prove (a) it suffices to assume that f is a radial function on B. As in [16], suppose $f(x)=u(|x|^2)$, where u is a C^2 function on $[0,1)$. Then with $r=|x|$,

$$\Delta f(x)=4r^2u''(r^2)+2nu'(r^2),\quad \text{and}\quad \langle x,\nabla f(x)\rangle=2r^2u'(r^2).$$

Thus by (3.2)

$$\widetilde{\Delta}f(x)=4r^2(1-r^2)^2u''(r^2)+2n(1-r^2)^2u'(r^2)+4(n-2)r^2(1-r^2)u'(r^2).$$

If we let

$$v(t)=t^{\frac{n}{2}}(1-t)^{2-n}u'(t),$$

then $\widetilde{\Delta}f(x)=4r^{2-n}(1-r^2)^nv'(r^2)$. Therefore

$$\begin{aligned}\int_{B_\rho}\widetilde{\Delta}f\,d\tau&=n\int_0^\rho r^{n-1}(1-r^2)^{-n}\widetilde{\Delta}f(r)\,dr\\&=4n\int_0^\rho rv'(r^2)\,dr\\&=2n\rho^n(1-\rho^2)^{2-n}u'(\rho^2).\end{aligned}$$

On the other–hand,

$$\frac{d}{d\rho}\int_S f(\rho\zeta)\,d\sigma(\zeta)=2\rho u'(\rho^2).$$

Therefore

$$\frac{d}{d\rho}\int_S f(\rho\zeta)\,d\sigma(\zeta)=\frac{1}{n}\rho^{1-n}(1-\rho^2)^{n-2}\int_{B_\rho}\widetilde{\Delta}f(x)\,d\tau(x),$$

which proves (a). Integrating the above from 0 to r gives

$$\int_S f(r\zeta)d\sigma(\zeta)-f(0)=\int_0^r\frac{1}{n}\rho^{1-n}(1-\rho^2)^{n-2}\int_{B_\rho}\widetilde{\Delta}f(x)\,d\tau(x)\,d\rho,$$

which upon changing the order of integration

$$=\int_{B_r}g(|x|,r)\widetilde{\Delta}f(x)\,d\tau(x),$$

where $g(|x|,r)$ is defined by (4.2). $\square$

Remark. Although we assumed in Theorem 4.2 that $f\in C^2(B)$, the conclusions are still valid if $f\in C^2(\Omega)$, where Ω is an open subset of B with $0\in\Omega$. In this case the conclusions hold for all r such that $B(0,r)\subset\Omega$.

As a corollary of Theorem 4.2 we obtain the following mean-value property for $\mathcal{M}$–harmonic and $\mathcal{M}$–subharmonic functions.

Corollary 4.3. (Invariant Mean–Value Property) *Let Ω be an open subset of B. A C^2 function f is $\mathcal{M}$–subharmonic on Ω if and only if for all $a \in \Omega$,*

$$f(a) \le \int_S f(\varphi_a(rt))\, d\sigma(t) \tag{4.3}$$

for all $r > 0$ such that $E(a, r) \subset \Omega$. Furthermore, f is $\mathcal{M}$–harmonic on B if and only if equality holds in (4.3).

Proof. Suppose f is $\mathcal{M}$–subharmonic on Ω and $a \in \Omega$. Then $f \circ \varphi_a$ is $\mathcal{M}$–subharmonic on $\varphi_a(\Omega)$. Since $\widetilde{\Delta}(f \circ \varphi_a) \ge 0$ on $\varphi_a(\Omega)$, by part (b) of the previous theorem we have

$$f(a) = (f \circ \varphi_a)(0) \le \int_S (f \circ \varphi_a)(rt)\, d\sigma(t)$$

for all $r > 0$ such that $B(0, r) \subset \varphi_a(\Omega)$. Clearly, if f is $\mathcal{M}$–harmonic on B, then equality holds in (4.3).

Conversely, suppose $f \in C^2(\Omega)$ satisfies (4.3) for all $a \in \Omega$ and $r > 0$ such that $E(a, r) \subset \Omega$. Let $a \in \Omega$ be arbitrary, and set $h(x) = (f \circ \varphi_a)(x)$. Then there exists $r_o > 0$ such that

$$h(0) \le \int_S h(rt) d\sigma(t)$$

for all r, $0 < r < r_o$. Since h is C^2 in a neighborhood of 0, as a consequence of the Taylor expansion of h about 0,

$$\int_S \{h(rt) - h(0)\}\, d\sigma(t) = \frac{r^2}{2n} \Delta h(0) + O(r^3). \tag{4.4}$$

Therefore,

$$\Delta h(0) = \lim_{r \to 0} \frac{2n}{r^2} \int_S \{h(rt) - h(0)\}\, d\sigma(t),$$

and thus $\widetilde{\Delta} f(a) = \Delta h(0) \ge 0$. Hence f is $\mathcal{M}$–subharmonic on B. □

Remarks. (a) If $f \in C^2(B)$ is $\mathcal{M}$–subharmonic on B, then Inequality (4.3) holds for all r, $0 < r < 1$. Furthermore, the integral mean

$$M(f, r) = \int_S f(rt) d\sigma(t)$$

is a non-decreasing function of r, $0 < r < 1$.

(b) As a consequence of the above proof, for $f \in C^2(\Omega)$, we have

$$\widetilde{\Delta} f(a) = \lim_{r \to 0} \frac{2n}{r^2} \int_S \{f(\varphi_a(rt)) - f(a)\} d\sigma(t). \tag{4.5}$$

One other consequence of Theorem 4.2 is the following corollary.

Corollary 4.4. *If $f \in C_c^2(B)$, then for all $a \in B$,*

$$f(a) = -\int_B G(a,x)\widetilde{\Delta} f(x) d\tau(x),$$

where G is the Green's function for B.

Proof. Since f has compact support in B, letting $r \to 1$ in Theorem 4.2 (b) gives

$$f(0) = -\int_B g(|x|)\widetilde{\Delta} f(x) d\tau(x).$$

Applying this to $f \circ \varphi_a$ gives the desired result. □

When $n = 2$, $d\tau(z) = \frac{1}{\pi}(1-|z|^2)^{-2} dA(z)$. Thus Corollary 4.4 simply becomes

$$f(z) = -\frac{1}{2\pi}\int_{\mathbb{D}} \log\left|\frac{z-w}{1-\overline{w}z}\right| \Delta f(w)\, dA(w)$$

for all $f \in C^2(\mathbb{D})$ with compact support. We conclude this section with the following version of the maximum principle.

Theorem 4.5. (Maximum Principle) *Suppose Ω is an open subset of B and that $f \in C^2(\Omega)$ is subharmonic in Ω and continuous on $\overline{\Omega}$. If $f \le 0$ on $\partial\Omega$, then $f \le 0$ in Ω.*

Proof. Set $h(x) = f(x) + \epsilon|x|^2$. Then $h \le \epsilon$ on $\partial\Omega$ and for all $x \in \Omega$,

$$\widetilde{\Delta} h(x) = \widetilde{\Delta} f(x) + 2\epsilon[n(1-|x|^2)^2 + 2(n-2)|x|^2(1-|x|^2)].$$

Thus $\widetilde{\Delta} h(x) > 0$ for all $x \in \Omega$. If h had a local maximum at some point $x \in \Omega$, then $h \circ \varphi_x$ has a local maximum at 0. This however is impossible since

$$\Delta(h \circ \varphi_x)(0) = \widetilde{\Delta} h(x) > 0.$$

Thus $h(x) < \epsilon$ for all $x \in \Omega$. Finally, since $f(x) \le h(x)$ for all x, letting $\epsilon \to 0$ gives $f(x) \le 0$ on Ω. □

4.2. $\mathcal{M}$–Subharmonic Functions

We now extend our definition of $\mathcal{M}$–subharmonic functions to the class of upper semicontinuous functions.

Definition 4.6. *Let Ω be an open subset of B. An upper semicontinuous function $f : \Omega \mapsto [-\infty, \infty)$, with $f \not\equiv -\infty$, is $\mathcal{M}$–subharmonic on Ω if*

$$f(a) \leq \int_S f(\varphi_a(rt))\, d\sigma(t) \tag{4.6}$$

for all $a \in \Omega$ and all r sufficiently small. A function f is $\mathcal{M}$–superharmonic if $-f$ is $\mathcal{M}$-subharmonic.

Inequality (4.6) is the Möbius invariant mean–value inequality. Definition 4.6 is of course equivalent to Definition 4.1 for the class of C^2 functions. In the following theorem we prove that with this definition the class of $\mathcal{M}$–subharmonic functions is again invariant under $\mathcal{M}$.

Theorem 4.7. *If f is $\mathcal{M}$–subharmonic on B, then $f \circ \psi$ is $\mathcal{M}$-subharmonic for all $\psi \in \mathcal{M}(B)$.*

Proof. For $\psi \in \mathcal{M}$ and $a \in B$, let $b = \psi(a)$. Then $(\varphi_b \circ \psi \circ \varphi_a)(0) = 0$, and thus $\psi(\varphi_a(x)) = \varphi_b(Ax)$ for some $A \in O(n)$. Consequently, by the $O(n)$ invariance of σ,

$$\begin{aligned}\int_S (f \circ \psi)(\varphi_a(rt))\, d\sigma(t) &= \int_S f(\varphi_b(Art))\, d\sigma(t)\\ &= \int_S f(\varphi_b(rt))\, d\sigma(t) \geq f(b) = (f \circ \psi)(a). \quad \square\end{aligned}$$

There is also a volume version of the invariant mean–value inequality.

Theorem 4.8. *Let Ω be an open subset of B. If f is $\mathcal{M}$–subharmonic on Ω, then for each $a \in \Omega$,*

$$f(a) \leq \frac{1}{\tau(B_r)} \int_{E(a,r)} f(x) d\tau(x) \tag{4.7}$$

for all r sufficiently small such that $E(a, r) \subset \Omega$. If f is $\mathcal{M}$–harmonic on Ω, then equality holds in (4.7).

Proof. The inequality follows from the $\mathcal{M}$–invariance of τ, and is obtained by multiplying both sides of (4.6) by $n\,\rho^{n-1}(1-\rho^2)^{-n}$ and integrating from 0 to r. $\square$

Remark. If f is $\mathcal{M}$–harmonic on B, then

$$f(\psi(0)) = \int_S f(\psi(rt))\, d\sigma(t)$$

for all $\psi \in \mathcal{M}(B)$. By multiplying by $n\rho^{n-1}$ and integrating we also have

$$f(\psi(0)) = \int_B f(\psi(w))\, d\nu(w) \tag{4.8}$$

for all $\psi \in \mathcal{M}(B)$. This leads to the following question.

Question: If $f \in L^1(B)$ satisfies (4.8) for every $\psi \in \mathcal{M}(B)$, is f $\mathcal{M}$–harmonic?

It is known that the answer is yes when $n = 2$. On the hermitian hyperbolic ball B in $\mathbb{C}^m$ the answer to the analogous question is very surprising. In [1] P. Ahern, M. Flores, and W. Rudin answered the question in the affirmative when $m \le 11$, and in the negative when $m \ge 12$.

Since many of the properties of subharmonic functions follow from the euclidean analogue of (4.6) or (4.7), those same properties are still valid for $\mathcal{M}$–subharmonic functions on B. In particular, $\mathcal{M}$–subharmonic functions are bounded above on compact subsets of B and are locally integrable on B, i.e., $\int_K f\, d\tau > -\infty$ for every compact set K. Also, $\mathcal{M}$–subharmonic functions satisfy the following version of the maximum principle.

Theorem 4.9. (Maximum Principle) *Let Ω be an open connected subset of B. If f is a non-constant $\mathcal{M}$-subharmonic function on Ω, then*

$$f(x) < \sup_{y\in\Omega} f(y), \qquad \textit{for all} \quad y \in \Omega.$$

Proof. The proof of both the maximum principle and local integrability are the same as in the classical case using inequality (4.7) and a connectivity argument. □

For C^2 $\mathcal{M}$–subharmonic functions f there is also an invariant volume version of (4.5). If we set $h(x) = (f \circ \varphi_a)(x)$ and integrate (4.4), then for all r sufficiently small,

$$\int_{B_r} \{h(x) - h(0)\}\, d\tau(x) = \tfrac{1}{2}\Delta h(0)c(r) + O(r^{n+3}),$$

where

$$c(r) = \int_0^r \rho^{n+1}(1-\rho^2)^{-n} d\rho.$$

By L'Hospital's rule,

$$\lim_{r\to 0} \frac{r^2\tau(B_r)}{c(r)} = n + 2 \lim_{r\to 0}(1-r^2)\left\{\frac{\tau(B_r)}{r^n/(1-r^2)^{n-1}}\right\},$$

which by (3.14)

$$= n + 2.$$

Also, since $r^2\tau(B_r) \approx r^{n+2}$ for r sufficiently small, we have

$$\widetilde{\Delta} f(a) = \lim_{r\to 0} \frac{2(n+2)}{r^2\tau(B_r)} \int_{E(a,r)} \{f(x) - f(a)\}\, d\tau(x). \tag{4.9}$$

4.3. The Invariant Convolution on B

For $0 < p < \infty$, we denote by $L^p(B,\tau)$ the space of measurable functions f on B for which

$$\|f\|_p^p = \int_B |f(x)|^p \, d\tau(x) < \infty.$$

Also, $L^p_{loc}(B)$ will denote the space of measurable functions f on B that are locally p-integrable, i.e.,

$$\int_K |f(x)|^p \, d\tau(x) < \infty$$

for every compact subset K of B.

Definition 4.10. *For measurable functions f, g on B, the invariant convolution $f * g$ of f and g is defined by*

$$(f * g)(y) = \int_B f(x) g(\varphi_y(x)) \, d\tau(x), \qquad y \in B,$$

provided this integral exists.

By the invariance of τ we have $(f*g)(y) = (g*f)(y)$. Although the convolution as defined is not the usual definition for convolution of functions on a topological group, the following analogue of the standard convolution inequalities are still valid. The proofs of the following two lemmas are identical to the proofs of the corresponding results for hermitian hyperbolic space and thus are omitted. Details may be found in [21, Section 4.2].

Lemma 4.11. *Let $p \in [1, +\infty)$ and let p' be defined by $\frac{1}{p} + \frac{1}{p'} = 1$. If $f \in L^p(B,\tau)$, then*

$$\|f * g\|_p \le \|f\|_p \|g\|_1$$

for all radial functions $g \in L^1(B,\tau)$, and

$$\|f * g\|_\infty \le \|f\|_p \|g\|_{p'}$$

for all radial functions $g \in L^{p'}(B,\tau)$.

As a consequence of Lemma 4.11, if $g \in L^1_{loc}(B)$ is radial and $f \in L^p_{loc}(B)$ then $f*g$ is defined a.e. on B. There is one additional property of the above convolution that will be needed.

Lemma 4.12. *If f, χ, $h \in L^1(B,\tau)$ and χ is radial, then $(f * \chi) * h = f * (\chi * h)$.*

We now express Identity (4.9) in terms of the above convolution. If we define Ω_r by

$$\Omega_r(x) = \begin{cases} \dfrac{1}{\tau(B_r)} & |x| \le r, \\ 0 & |x| > r, \end{cases}$$

then

$$(f * \Omega_r)(a) - f(a) = \frac{1}{\tau(B_r)} \int_{E(a,r)} \{f(x) - f(a)\}\, d\tau(x).$$

Thus by (4.9), if $f \in C^2(B)$, the

$$(\widetilde{\Delta} f)(a) = \lim_{r \to 0} \frac{2(n+2)}{r^2} \left[(f * \Omega_r)(a) - f(a)\right]. \tag{4.10}$$

If f has compact support, then the convergence is uniform on B.

4.4. Approximation by C^∞ $\mathcal{M}$–Subharmonic Functions

Definition 4.13. *Let $\{r_k\}$ be a decreasing sequence with $r_k \to 0$ as $k \to \infty$. For each k, let χ_k be a non–negative C^∞ radial function on B with support contained in $\{x : r_{k+1} < |x| < r_k\}$ satisfying*

$$\int_B \chi_k(x)\, d\tau(x) = 1.$$

The sequence $\{\chi_k\}_{k=1}^\infty$ is a C^∞ approximate identity for $L^1(B, \tau)$.

The following lemma, whose proof follows from the usual arguments, is stated without proof.

Lemma 4.14. *Let $\{\chi_k\}$ be defined as above. Then*

$$\lim_{k \to \infty} (h * \chi_k) = h,$$

uniformly on B if $h \in C_c(B)$, and locally in L^p if $h \in L^p_{loc}(B)$.

We are now in a position to prove the following theorem.

Theorem 4.15. *Let $\{\chi_k\}$ be a C^∞ approximate identity as above. If f is $\mathcal{M}$-subharmonic on B, then $\{f * \chi_k\}_{k=1}^\infty$ is a non–increasing sequence of C^∞ $\mathcal{M}$–subharmonic functions on B satisfying*

$$(f * \chi_k)(x) \geq f(x) \quad \textit{and} \quad \lim_{k \to \infty} (f * \chi_k)(x) = f(x) \tag{4.11}$$

for all $x \in B$.[8]

---

[8]If f is a continuous function on B that is both $\mathcal{M}$–subharmonic and $\mathcal{M}$–superharmonic on B, then $(f * \chi_k)(x) = f(x)$ for all $x \in B$. Thus f is C^∞ and satisfies $\widetilde{\Delta} f(x) = 0$ for all $x \in B$.

Proof. Since χ_k is C^∞, the function $f * \chi_k$ is also C^∞. By integration in polar coordinates,

$$\begin{aligned}(f * \chi_k)(y) &= n \int_0^1 \frac{r^{n-1}}{(1-r^2)^n} \chi_k(r) \left[\int_S f(\varphi_y(rt))\, d\sigma(t) \right] dr \\ &\geq f(y) \int_B \chi_k \, d\tau = f(y).\end{aligned}$$

Fix $a \in B$, and let $\alpha > f(a)$. Since f is upper semicontinuous there exists $r > 0$ such that $f(w) < \alpha$ for all $w \in E(a, r)$. Hence if $r_k < r$,

$$(f * \chi_k)(a) = \int_B f(w) \chi_k(\varphi_a(w))\, d\tau(w) \leq \alpha \int_B \chi_k \, d\tau = \alpha.$$

Thus $\limsup_{k\to\infty} (f * \chi_k)(a) \leq f(a)$, which when combined with the above proves (4.11).

To show that $f * \chi_k$ is $\mathcal{M}$-subharmonic we use (4.10). Since χ_k and Ω_r are both radial, by Lemma 4.12

$$(f * \chi_k) * \Omega_r = (f * \Omega_r) * \chi_k.$$

But as above, by integration in polar coordinates $(f * \Omega_r) \geq f$. Therefore

$$(f * \chi_k) * \Omega_r = (f * \Omega_r) * \chi_k \geq f * \chi_k.$$

Thus by (4.10) $\widetilde{\Delta}(f * \chi_k) \geq 0$. Therefore $f * \chi_k$ is $\mathcal{M}$–subharmonic.

Finally it remains to be shown that the sequence $\{f * \chi_k\}$ is non-increasing. For the proof we require that an $\mathcal{M}$–subharmonic function satisfy

$$\int_S f(rt)\, d\sigma(t) \leq \int_S f(\rho t)\, d\sigma(t) \tag{4.12}$$

whenever $0 < r < \rho < 1$. By the remark following the proof of Corollary 4.3, inequality (4.12) is valid whenever f is a C^2 $\mathcal{M}$–subharmonic function. Thus since $f \leq f * \chi_k$,

$$\int_S f(rt) d\sigma(t) \leq \int_S (f * \chi_k)(rt) d\sigma(t) \leq \int_S (f * \chi_k)(\rho t) d\sigma(t).$$

But since subharmonic functions are locally integrable and bounded above on compact sets, we have

$$\limsup_{k\to\infty} \int_S (f * \chi_k)(\rho t) d\sigma(t) \leq \int_S f(\rho t) d\sigma(t),$$

which proves (4.12).

Since $(f * \chi_k)(a) = ((f \circ \varphi_a) * \chi_k)(0)$, it suffices to prove that the sequence $\{(f * \chi_k)(0)\}$ is non–increasing. Suppose $m > k$. Since the support of χ_k is contained in $\{r_{k+1} < |x| < r_k\}$ and $r_{k+1} \ge r_m$, we have

$$\begin{aligned}
(f * \chi_k)(0) &= n \int_{r_{k+1}}^{r_k} r^{n-1}(1-r^2)^{-n}\chi_k(r) \int_S f(rt)\, d\sigma(t)\, dr \\
&\ge \int_S f(r_m t)\, d\sigma(t) \int_B \chi_k \, d\tau \\
&= \int_S f(r_m t)\, d\sigma(t) \int_B \chi_m \, d\tau \\
&\ge n \int_{r_{m+1}}^{r_m} r^{n-1}(1-r^2)^{-n}\chi_m(r) \int_S f(rt)\, d\sigma(t)\, dr \\
&= (f * \chi_m)(0). \quad \square
\end{aligned}$$

4.5. The Weak Laplacian and Riesz Measure

Out goal in this section is to provide a characterization of $\mathcal{M}$–subharmonic functions in terms of the "weak Laplacian". For this we require Green's identity (see[18]) for the invariant Laplacian. If $f,\, g \in C^2(B)$ and one of them has compact support, then

$$\int_B f \widetilde{\Delta} g \, d\tau = \int_B g \widetilde{\Delta} f \, d\tau.$$

Thus if f is a C^2 $\mathcal{M}$–subharmonic function, we have $\int_B f \widetilde{\Delta}\psi \, d\tau \ge 0$ for all $\psi \in C_c^2(B)$ with $\psi \ge 0$. Thus for for $f \in L^1_{loc}$ we say that $\widetilde{\Delta} f \ge 0$ in the *weak sense* if $\int f \widetilde{\Delta}\psi d\tau \ge 0$ for all $\psi \in C_c^2(B)$ with $\psi \ge 0$.

Theorem 4.16. *If f is $\mathcal{M}$–subharmonic on B, then*

$$\int_B f(z) \widetilde{\Delta}\psi(z)\, d\tau(z) \ge 0 \tag{4.13}$$

for all $\psi \in C_c^2(B)$ with $\psi \ge 0$. Conversely, if $f \in L^1_{loc}(B)$ is such that (4.13) holds for all $\psi \in C_c^2(B)$ with $\psi \ge 0$, then there exists an $\mathcal{M}$–subharmonic function F on B such that $F = f$ a.e. on B.

Proof. Let $\{\chi_k\}$ be a C^∞ approximate identity as given in Definition 4.13. Suppose f is $\mathcal{M}$–subharmonic on B. Set $f_k = f * \chi_k$. Then by Theorem 4.15, $\{f_k\}$ is a non–increasing sequence of C^∞ $\mathcal{M}$–subharmonic functions on B that converges to f everywhere on B. Thus by Green's identity and the monotone convergence theorem,

$$\begin{aligned}\int_B f\,\widetilde{\Delta}\psi\,d\tau &= \lim_{k\to\infty}\int_B f_k\,\widetilde{\Delta}\psi\,d\tau\\ &= \lim_{k\to\infty}\int_B \psi\,\widetilde{\Delta}f_k\,d\tau \geq 0\end{aligned}$$

for all $\psi \in C_c^2(B)$ with $\psi \geq 0$. This proves (4.13).

Conversely, suppose $f \in L^1_{loc}(B)$ satisfies (4.13). Let f_k be defined as above. In the notation of convolutions, hypothesis (4.13) is just

$$(f * \widetilde{\Delta}\psi)(0) \geq 0$$

for all $\psi \in C_c^2(B)$ with $\psi \geq 0$. It follows from the definition that

$$\widetilde{\Delta}f_k(a) = \widetilde{\Delta}(f * \chi_k)(a) = (f * \widetilde{\Delta}(\chi_k \circ \varphi_a))(0).$$

Thus since $\chi_k \circ \varphi_a$ is C^∞ with compact support, $\widetilde{\Delta}f_k(a) \geq 0$. Therefore f_k is $\mathcal{M}$–subharmonic on B.

We now show that the sequence $\{f_k\}$ is non–increasing. Suppose $k > m$ and j is arbitrary. Since f_j is $\mathcal{M}$–subharmonic, we have

$$\begin{aligned}f_m * \chi_j = (f * \chi_m) * \chi_j &= (f * \chi_j) * \chi_m\\ &\geq (f * \chi_j) * \chi_k = f_k * \chi_j.\end{aligned}$$

Since f_m, f_k are continuous, $\lim\limits_{j\to\infty} f_m * \chi_j = f_m$, with a similar result for f_k. Thus the sequence $\{f_k\}$ is non–increasing. Define

$$F(x) = \lim_{k\to\infty} f_k(x)$$

which exists everywhere on B. As a consequence of the mean–value inequality, F is either $\mathcal{M}$-subharmonic on B, or $F \equiv -\infty$. But by Lemma 4.14, $\{f_k\}$ converges to f locally in L^1, and thus $F = f$ a.e. on B. $\square$

Theorem 4.17. *If f is $\mathcal{M}$–subharmonic on B, then there exists a unique regular Borel measure μ_f on B such that*

$$\int_B \psi\,d\mu_f = \int_B f\widetilde{\Delta}\psi\,d\tau \tag{4.14}$$

for all $\psi \in C_c^2(B)$.

Definition 4.18. *If f is $\mathcal{M}$–subharmonic on B, the unique regular Borel measure μ_f satisfying (4.14) is called the Riesz measure of f.*

Proof. Let f be $\mathcal{M}$–subharmonic on B. By (4.13),

$$L(\psi) = \int_B f\, \widetilde{\Delta}\psi\, d\tau$$

defines a non–negative linear functional on $C_c^\infty(B)$. We extend L to $C_c(B)$ as follows. Let $\psi \in C_c(B)$. Choose a sequence $\{\psi_k\} \subset C_c^\infty(B)$ such that $\psi_k \to \psi$ uniformly on B. Choose a compact subset K of B such that the support of ψ and ψ_k, $k = 1, 2...$ are contained in K. Let V be a relatively compact open subset of B such that $K \subset V$, and let $h \in C_c^\infty(B)$, $0 \le h \le 1$, be such that $h \equiv 1$ on K and the support of h is contained in V. Set

$$\epsilon_{k,m} = \sup_{x \in K} |\psi_k(x) - \psi_m(x)|.$$

Then for all $x \in B$,

$$-\epsilon_{k,m} h(x) \le \psi_k(x) - \psi_m(x) \le \epsilon_{k,m} h(x).$$

Thus since L is positive,

$$|L(\psi_k) - L(\psi_m)| \le \epsilon_{k,m}\, L(h).$$

Therefore $\{L(\psi_k)\}$ is Cauchy. Define

$$L(\psi) = \lim_{k \to \infty} L(\psi_k).$$

It is easy to show that $L(\psi)$ is independent of the choice of $\{\psi_k\}$, and thus defines a non–negative linear functional on $C_c(B)$. The result now follows by the Riesz representation theorem for non–negative linear functionals on $C_c(B)$. $\square$

5. The Poisson Kernel and Poisson Integrals

5.1. The Poisson Kernel for $\widetilde{\Delta}$

In this section we provide a heuristic argument for the formula for the Poisson kernel on B. We begin by stating the *Green's formula* (see [18, (92.5)]) for the invariant Laplacian: if Ω is an open subset of B, $\overline{\Omega} \subset B$, whose boundary is sufficiently smooth, then if $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$,

$$\int_{\Omega} (u\widetilde{\Delta}v - v\widetilde{\Delta}u)d\tau = \int_{\partial\Omega} (uD_{\tilde{n}}v - vD_{\tilde{n}}u)\ d\tilde{\sigma}, \tag{5.15}$$

where $\tilde{\sigma}$ is the surface element on $\partial\Omega$ with respect to the hyperbolic metric, and $D_{\tilde{n}}$ denotes the normal derivative in the outward normal direction again with respect to the hyperbolic metric. With $\Omega = B_r$, the above becomes

$$\int_{B_r} (u\widetilde{\Delta}v - v\widetilde{\Delta}u)d\tau = \int_{S_r} (uD_{\tilde{n}}v - vD_{\tilde{n}}u)\ d\tilde{\sigma}.$$

For the surface S_r given by $f(x) = 0$, where $f(x) = |x|^2 - r^2$, the normal derivative $D_{\tilde{n}}$ of v is given by

$$D_{\tilde{n}}v(r\zeta) = \frac{\langle \widetilde{\nabla}v(r\zeta), \widetilde{\nabla}f(r\zeta)\rangle}{|\widetilde{\nabla}f(r\zeta)|} = (1 - r^2)\langle \nabla v(r\zeta), \zeta\rangle.$$

Also, by Theorem 4.2 (a),

$$\begin{aligned}
\int_{B_r} \widetilde{\Delta}u(x)d\tau(x) &= nr^{n-1}(1 - r^2)^{2-n}\frac{d}{dr}\int_S f(r\zeta)\, d\sigma(\zeta) \\
&= nr^{n-1}(1 - r^2)^{2-n}\int_S \langle \nabla u(r\zeta), \zeta\rangle\, d\sigma(\zeta) \\
&= nr^{n-1}(1 - r^2)^{1-n}\int_S D_{\tilde{n}}u(r\zeta)\, d\sigma(\zeta).
\end{aligned}$$

Thus the surface element $d\tilde{\sigma}$ on S_r is given by

$$d\tilde{\sigma}(r\zeta) = \frac{nr^{n-1}}{(1 - r^2)^{n-1}}\, d\sigma(\zeta). \tag{5.16}$$

Suppose now that u is $\mathcal{M}$–harmonic on B, and for the sake of simplicity C^1 on $\overline{B}$. Let $a \in B$ be arbitrary, and let $r_o > 0$ be such that $a \in B_r$ for all $r \geq r_o$. Choose $\epsilon > 0$ such that $E(a,\epsilon) \subset B_{r_o}$. Then by Green's formula applied to $\Omega_\epsilon = B_r \backslash E(a,\epsilon)$ with $v(x) = G(a,x)$, and the $\mathcal{M}$–invariance of $\tilde{\sigma}$,

$$\int_{S_r} \{uD_{\tilde{n}}v - vD_{\tilde{n}}u\}\, d\tilde{\sigma} = \int_{S_\epsilon} \{(u\circ\varphi_a)D_{\tilde{n}}g - gD_{\tilde{n}}(u\circ\varphi_a)\}\, d\tilde{\sigma},$$

where g is given by (3.5). Clearly $\lim\limits_{\epsilon\to 0}\int_{S_\epsilon} gD_{\tilde{n}}(u\circ\varphi_a)d\tilde{\sigma} = 0$. On the other–hand, since g is radial,

$$D_{\tilde{n}}g(\epsilon\zeta) = (1-\epsilon^2)g'(\epsilon) = -\frac{(1-\epsilon^2)^{n-1}}{n\epsilon^{n-1}}.$$

Thus by (5.16)

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{S_\epsilon}(u\circ\varphi_a)D_{\tilde{n}}g\, d\tilde{\sigma} &= -\lim_{\epsilon\to 0}\int_S u(\varphi_a(\epsilon\zeta))\, d\sigma(\zeta)\\ &= -u(a).\end{aligned}$$

Hence for any $r > r_o$,

$$u(a) = -\int_{S_r}[uD_{\tilde{n}}v - vD_{\tilde{n}}u]\, d\tilde{\sigma}.$$

Since $G(a,r) \approx (1-r^2)^{n-1}$ and $u \in C^1(\overline{B})$,

$$\lim_{r\to 1}\int_{S_r} vD_{\tilde{n}}u\, d\tilde{\sigma} \approx \lim_{r\to 1}(1-r^2)\int_S \langle\nabla u(r\zeta),\zeta\rangle\, d\sigma(\zeta) = 0.$$

Thus setting $G_a(x) = G(a,x)$,

$$\begin{aligned}u(a) &= -\lim_{r\to 1}\int_{S_r} uD_{\tilde{n}}G_a d\tilde{\sigma}\\ &= -\lim_{r\to 1} nr^{n-1}(1-r^2)^{2-n}\int_S u(r\zeta)\langle\nabla G_a(r\zeta),\zeta\rangle\, d\sigma(\zeta)\\ &= \int_S P(a,\zeta)u(\zeta)\, d\sigma(\zeta),\end{aligned}$$

where

$$P(a,\zeta) = -\lim_{r\to 1} nr^{n-1}(1-r^2)^{2-n}\langle\nabla G_a(r\zeta),\zeta\rangle. \tag{5.17}$$

Our next step is to compute the above limit. Since $G_a(x) = g(|\varphi_a(x)|)$, where g is the radial function defined by (3.5),

$$\begin{aligned}\nabla G_a(x) &= -\frac{(1-|\varphi_a(x)|^2)^{n-2}}{n|\varphi_a(x)|^{n-1}}\,\nabla|\varphi_a(x)|\\ &= -\frac{(1-|a|^2)^{n-2}(1-|x|^2)^{n-2}}{n|x-a|^{n-1}\rho(x,a)^{\frac{1}{2}(n-3)}}\,\nabla|\varphi_a(x)|.\end{aligned}$$

Using Identity (2.6) we have

$$2|\varphi_a(x)|\nabla|\varphi_a(x)| = \nabla\left[\frac{|x-a|^2}{\rho(x,a)}\right].$$

From this it now follows that

$$\nabla|\varphi_a(x)| = \frac{(x-a)(1-|x|^2)(1-|a|^2) + x|x-a|^2(1-|a|^2)}{|x-a|\rho(x,a)^{\frac{3}{2}}}.$$

Therefore

$$\begin{aligned} &-nr^{n-1}(1-r^2)^{2-n}\langle\nabla G_a(r\zeta),\zeta\rangle \\ &\quad= \frac{r^{n-1}(1-|a|^2)^{n-2}}{|r\zeta-a|^n\rho(r\zeta,a)^{\frac{n}{2}}}\left[(1-r^2)(1-|a|^2)\langle r\zeta-a,\zeta\rangle + r|r\zeta-a|^2(1-|a|^2)\right]. \end{aligned}$$

Thus

$$-\lim_{r\to 1} r^{n-1}(1-r^2)^{2-n}\langle\nabla G_a(r\zeta),\zeta\rangle = \frac{(1-|a|^2)^{n-1}|\zeta-a|^2}{|\zeta-a|^n\rho(\zeta,a)^{\frac{n}{2}}},$$

which since $\rho(\zeta,a) = |\zeta-a|^2$ gives

$$P(a,\zeta) = \left(\frac{1-|a|^2}{|\zeta-a|^2}\right)^{n-1}. \tag{5.18}$$

Definition 5.19. *The function P on $B\times S$ defined by (5.18) is called the invariant Poisson kernel for $\widetilde{\Delta}$ on B.*

A tedious, but straightforward computation proves that for each $t \in S$, the function $x \mapsto P(x,t)$ is $\mathcal{M}$–harmonic on B.

In contrast to the above, the Poisson kernel $\mathcal{P}$ for the ordinary Laplacian Δ on $B\times S$ is given by

$$\mathcal{P}(x,t) = \frac{1-|x|^2}{|x-t|^n}, \qquad (x,t)\in B\times S,$$

whereas the Poisson kernel $\widetilde{\mathcal{P}}$ for the invariant Laplacian $\widetilde{\Delta}$ on the hermitian hyperbolic ball B in $\mathbb{C}^n$ is given by

$$\widetilde{\mathcal{P}}(z,t) = \frac{(1-|z|^2)^n}{|1-\langle z,t\rangle|^{2n}}, \qquad (z,t)\in B\times S.$$

It is only in the unit disc $\mathbb{D}$ in $\mathbb{R}^2$ that all three agree.

5.2. The Dirichlet Problem for $\widetilde{\Delta}$

We summarize some of the properties of the Poisson kernel in the following lemma. These are analogous to the properties of the Poisson kernel for Δ on B.

Lemma 5.20. *The invariant Poisson kernel P on $B \times S$ satisfies the following:*

(a) *For fixed $t \in S$, $x \mapsto P(x,t)$ is $\mathcal{M}$–harmonic on B,*

(b) *$P(r\zeta, t) = P(rt, \zeta)$ for all $t, \zeta \in S$,*

(c) $\displaystyle\int_S P(x,t)\,d\sigma(t) = 1$, *and*

(d) *For fixed $\zeta \in S$ and $\delta > 0$,* $\displaystyle\lim_{\substack{x\to\zeta \\ x\in B}} \int_{|t-\zeta|>\delta} P(x,t)\,d\sigma(t) = 0.$

Proof. As indicated above, (a) follows by computation, and (b) is almost obvious. Writing $x = r\zeta$, (c) follows by (b) and the mean-value property for $\mathcal{M}$–harmonic functions. The proof of (d) is again standard. $\square$

Definition 5.21. *For $f \in L^1(S)$, the Poisson integral of f denoted $P[f]$ is defined by*

$$P[f](x) = \int_S P(x,t)f(t)\,d\sigma(t).$$

Similarly, if μ is a finite Borel measure on S, the Poisson integral of μ will be denoted by $P[\mu]$.

Since the function $t \mapsto P(x,t)$ is continuous on S, the above integrals exist and are finite for all $x \in B$. Furthermore, as a consequence of the mean-value property, the function $P[\mu](x)$ is $\mathcal{M}$–harmonic on B. The results (c) and (d) of Lemma 5.20 imply that the Poisson kernel behaves as an approximate identity. The usual methods of proof now give that for $f \in L^1(S)$,

$$\lim_{\substack{x\to\zeta \\ x\in B}} P[f](x) = f(\zeta) \tag{5.19}$$

at each $\zeta \in S$ where f is continuous. This then proves part (a) of the following theorem.

Theorem 5.22. **(a)** *If $f \in C(S)$, then the function F defined by*

$$F(x) = \begin{cases} P[f](x), & x \in B, \\ f(x), & x \in S, \end{cases}$$

is $\mathcal{M}$–harmonic on B and continuous on $\overline{B}$ with $\|F\|_\infty = \|f\|_\infty$.

(b) *Conversely, if f is $\mathcal{M}$–harmonic on B and continuous on $\overline{B}$, then $f(x) = P[f](x)$.*

Proof. The proof of (b) is an immediate consequence of the maximum principle (Theorem 4.5) applied to $F(x) = f(x) - P[f](x)$. $\square$

An immediate consequence of the previous is the following analogue of Theorem 3.3.8 of [17].

Theorem 5.23. *If $f \in L^1(S)$, then $P[f \circ \psi] = P[f] \circ \psi$ for all $\psi \in \mathcal{M}(B)$.*

Proof. Since $C(S)$ is dense in $L^1(S)$, it suffices to prove the result for continuous functions on S. If $f \in C(S)$, then by the previous theorem $F(x) = P[f](x)$ is $\mathcal{M}$–harmonic on B and continuous on $\overline{B}$. Suppose $\psi \in \mathcal{M}(B)$. Since ψ is continuous on $\overline{B}$, $F \circ \psi$ is also $\mathcal{M}$–harmonic on B, continuous on $\overline{B}$ with $\lim_{x \to \zeta}(F \circ \psi)(x) = f(\psi(\zeta))$ for all $\zeta \in S$. Thus $(F \circ \psi)(x) = P[f \circ \psi](x)$. On the other–hand, $(F \circ \psi)(x) = F(\psi(x)) = P[f](\psi(x))$, which proves the result. $\square$

If in the above proof we take $\psi = \varphi_a$ and $x = 0$, then

$$\int_S f(\varphi_a(t))\, d\sigma(t) = \int_S P(a,t) f(t)\, d\sigma(t) \tag{5.20}$$

for all $f \in L^1(S)$.

There is a significant difference between invariant Poisson integrals and solutions of the classical Dirichlet problem for the Laplacian Δ. Since Δ is uniformly elliptic, if $f \in C^\infty(S)$ and F is the euclidean Poisson integral of f, then $F \in C^\infty(\overline{B})$. The following example shows that this fails dramatically for invariant Poisson integrals.

Example 5.24. To illustrate the above we compute the invariant Poisson integral of the function $f(t) = t_1^2$ for $n = 3$. In this case

$$F(x) = (1 - |x|^2)^2 \int_S \frac{t_1^2}{|x - t|^4}\, d\sigma(t).$$

With $x = re_1$, where $e_1 = (1, 0, 0)$,

$$F(re_1) = (1 - r^2)^2 \int_S \frac{t_1^2}{(1 + r^2 - 2rt_1)^2}\, d\sigma(t),$$

which since the integrand is a function of t_1 only (see [2, p. 216])

$$\begin{aligned} &= \tfrac{1}{2}(1 - r^2)^2 \int_{-1}^{1} \frac{x^2}{(1 + r^2 - 2rx)^2}\, dx \\ &= \frac{1}{4r^3}\left[2r(1 + r^4) + (1 + r^2)(1 - r^2)^2 \log\left(\frac{1 - r}{1 + r}\right)\right]. \end{aligned}$$

Note, at $r = 0$, the term in brackets is of the form $\frac{4}{3}r^3 + O(r^5)$, and thus $F(re_1)$ is indeed continuous at 0. Even though $f(t) = t_1^2$ is C^∞ on S, the function $F(re_1)$ is not C^2 at the boundary point e_1. A formula valid for all $x \in B$ will be given in Example 5.27.

5.3. Spherical Harmonics

Throughout this section we will assume $n \geq 3$. The results for $n = 2$ are well known. As in [2, Chapter 5], for $m = 0, 1, 2, ..,$ we denote by $\mathcal{H}_m(\mathbb{R}^n)$ the space of all (euclidean) homogeneous harmonic polynomials of degree m on $\mathbb{R}^n$. A *spherical harmonic of degree* m is the restriction to S of a harmonic polynomial in $\mathcal{H}_m(\mathbb{R}^n)$. The collection of all spherical harmonic polynomials of degree m will be denoted by $\mathcal{H}_m(S)$. Every element of $\mathcal{H}_m(S)$ has a unique extension to $\mathcal{H}_m(\mathbb{R}^n)$. If $m \neq k$, then $\mathcal{H}_m(S)$ and $\mathcal{H}_k(S)$ are orthogonal in $L^2(S)$, i.e.,

$$\langle p, q\rangle = \int_S p(t)q(t)\, d\sigma(t) = 0$$

for all $p \in \mathcal{H}_m(S)$, $q \in \mathcal{H}_k(S)$. Furthermore,

$$L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S).$$

Our first goal is to solve the Dirichlet problem for $p_\alpha \in \mathcal{H}_\alpha(S)$. Since each p_α has a unique extension to $\mathcal{H}_\alpha(\mathbb{R}^n)$ we can assume that $p_\alpha(x) \in \mathcal{H}_\alpha(\mathbb{R}^n)$. Set $f(x) = g(r^2)p_\alpha(x)$, where $r^2 = |x|^2$. Since $\langle x, \nabla p_\alpha\rangle = \alpha p_\alpha(x)$, we have

$$\begin{aligned}\langle x, \nabla f(x)\rangle &= p_\alpha(x)\langle x, \nabla g\rangle + g(r^2)\langle x, \nabla p_\alpha\rangle \\ &= 2r^2 g'(r^2)p_\alpha(x) + \alpha g(r^2)p_\alpha(x),\end{aligned}$$

and since p_α is harmonic,

$$\begin{aligned}\Delta f(x) &= 2\langle \nabla p_\alpha, \nabla g\rangle + p_\alpha(x)\Delta g(r^2) \\ &= 4\alpha p_\alpha(x)g'(r^2) + p_\alpha(x)[2ng'(r^2) + 4r^2 g'(r^2)].\end{aligned}$$

Therefore

$$\begin{aligned}&\widetilde{\Delta} f(x) = 2(1-r^2)p_\alpha(x) \\ &\times \left[\ 2(1-r^2)r^2 g''(r^2) + \{(n+2\alpha)(1-r^2) + 2r^2(n-2)\}g'(r^2) + \alpha(n-2)g(r^2)\right].\end{aligned}$$

Thus in order that $\widetilde{\Delta} f(x) = 0$ we must have

$$2(1-r^2)r^2 g''(r^2) + \{(n+2\alpha)(1-r^2) + 2r^2(n-2)\}g'(r^2) + \alpha(n-2)g(r^2) = 0,$$

or

$$(1-r^2)r^2 g''(r^2) + \{(\alpha + \tfrac{1}{2}n) - (\alpha + 2 - \tfrac{1}{2}n)r^2\}g'(r^2) - \alpha(1 - \tfrac{1}{2}n)g(r^2) = 0.$$

If we set $t = r^2$, $a = \alpha$, $b = 1 - \frac{1}{2}n$, and $c = \alpha + \frac{1}{2}n$, then the above equation can be rewritten as

$$t(1-t)g''(t) + \{c - (a+b+1)t\}g'(t) - abg(t) = 0. \tag{5.21}$$

Equation (5.21) however is the hypergeometric equation, for which a particular solution is given by the hypergeometric function $F(a, b; c; t)$ [15], defined by

$$F(a, b; c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad |z| < 1 \tag{5.22}$$

In the above, $(a)_0 = 1$ and for $k = 1, 2, ...,$

$$(a)_k = a(a+1)\cdots(a+k-1).$$

If a is not a negative integer, then

$$(a)_k = \Gamma(a+k)/\Gamma(a),$$

where Γ is the Gamma function defined on $\mathbb{C} \setminus \{0, -1, -2, ...\}$. If $c - a - b > 0$, then the series in (5.22) converges absolutely for all z, $|z| \leq 1$. For the above values of a, b, and c, we have $c - a - b = n - 1$. Thus the function $g(r^2)$ is given by

$$g(r^2) = c_\alpha F(\alpha, 1 - \tfrac{1}{2}n; \alpha + \tfrac{1}{2}n; r^2), \tag{5.23}$$

for an arbitrary constant c_α. Define $S_{n,\alpha}(r)$ by

$$S_{n,\alpha}(r) = \frac{F(\alpha, 1 - \frac{1}{2}n; \alpha + \frac{1}{2}n; r^2)}{F(\alpha, 1 - \frac{1}{2}n; \alpha + \frac{1}{2}n; 1)}. \tag{5.24}$$

Then $S_{n,\alpha}(1) = 1$ and $f(x) = S_\alpha(|x|)p_\alpha(x)$ is a solution of $\widetilde{\Delta} f(x) = 0$ that is continuous on $\overline{B}$ with $f(\zeta) = p_\alpha(\zeta)$ for all $\zeta \in S$. This proves the following theorem.

Theorem 5.25. *If $p_\alpha \in \mathcal{H}_\alpha(S)$, $\alpha = 0, 1, 2, ...,$ then for all $t \in S$,*

$$P[p_\alpha](rt) = r^\alpha S_{n,\alpha}(r) p_\alpha(t),$$

where $S_{n,\alpha}$ is defined by (5.24).

Example 5.26. If n is even, say $n = 2m$, then $b = 1 - m$ and thus $(b)_k = 0$ for all $k \geq m$. Hence $g(r^2)$ is a polynomial of degree $n - 2$. When $n = 4$, $b = -1$ and

$$S_{4,\alpha}(r) = \tfrac{1}{2}(\alpha + 2)(1 - \tfrac{\alpha}{\alpha+2} r^2), \qquad \alpha = 0, 1, 2... .$$

When $n=6$, $b=-2$ and

$$F(\alpha,-2;\alpha+3;r^2)=1-\tfrac{2\alpha}{\alpha+3}r^2+\tfrac{\alpha(\alpha+1)}{(\alpha+3)(\alpha+4)}r^4.$$

Thus for $\alpha=0,1,2,...,$

$$S_{6,\alpha}(r)=\tfrac{1}{12}(\alpha+3)(\alpha+4)\left[1-\tfrac{2\alpha}{\alpha+3}r^2+\tfrac{\alpha(\alpha+1)}{(\alpha+3)(\alpha+4)}r^4\right].$$

By [15, Identity 9.3.4], if $c-a-b>0$ then

$$\lim_{t\to 1^-}F(a,b;c;t)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{5.25}$$

Therefore

$$F(\alpha,1-\tfrac{1}{2}n;\alpha+\tfrac{1}{2}n;1)=\frac{\Gamma(\alpha+\frac{1}{2}n)\Gamma(n-1)}{\Gamma(\frac{1}{2}n)\Gamma(\alpha+n-1)},$$

and hence $S_{n,\alpha}(r)=c_{n,\alpha}F(\alpha,1-\frac{1}{2}n;\alpha+\frac{1}{2}n;r^2)$, where

$$c_{n,\alpha}=\frac{\Gamma(\frac{1}{2}n)\Gamma(\alpha+n-1)}{\Gamma(\alpha+\frac{1}{2}n)\Gamma(n-1)}. \tag{5.26}$$

Also, using the transformation [15, Identity 9.5.3]

$$F(a,b;c;t)=(1-t)^{c-a-b}F(c-a,c-b;c;t),$$

we can express $S_{n,\alpha}(r)$ as

$$S_{n,\alpha}(r)=c_{n,\alpha}(1-r^2)^{n-1}F(\tfrac{1}{2}n,\alpha+n-1;\alpha+\tfrac{1}{2}n;r^2). \tag{5.27}$$

Theorem 5.25 can be used to compute the invariant Poisson integral of a polynomial q on S. By [2, Corollary 5.7], if q is a polynomial on $\mathbb{R}^n$ of degree m, then the restriction of q to S is a sum of spherical harmonics of degree at most m. That is, there exist $p_k\in\mathcal{H}_k(S)$, $k=0,1,...,m$, such that $q(t)=\sum_{k=0}^m p_k(t)$ for all $t\in S$. Hence

$$P[q](x)=\sum_{k=0}^m P[p_k](x).$$

But by Theorem 5.25, $P[p_k](x)=S_{n,k}(|x|)p_k(x)$. Thus

$$P[q](x)=\sum_{k=0}^m S_{n,k}(|x|)p_k(x).$$

The above computations are particularly easy when n is even. These computations are illustrated in the following examples.

Examples 5.27. **(a)** For our first example we consider the function $q(t) = t_1^2$ in $\mathbb{R}^4$. Then for $t \in S$, $q(t) = p_0(t) + p_2(t)$, where $p_0(x) = \frac{1}{4}$ and $p_2(x) = x_1^2 - \frac{1}{4}|x|^2$. Thus by Example 5.26, in $\mathbb{R}^4$

$$\begin{aligned} P[t_1^2](x) &= \tfrac{1}{4} + S_{4,2}(|x|)p_2(x) \\ &= \tfrac{1}{4} + (2 - |x|^2)(x_1^2 - \tfrac{1}{4}|x|^2). \end{aligned}$$

The above function is easily shown to be $\mathcal{M}$–harmonic on B.

(b) In spaces of odd dimension these computations are much more complicated. As in Example 5.24 consider $q(t) = t_1^2$ in $\mathbb{R}^3$. Then $q(t) = p_0(t) + p_2(t)$ where $p_0(x) = \frac{1}{3}$ and $p_2(x) = x_1^2 - \frac{1}{3}|x|^2$. Hence

$$P[q](x) = \tfrac{1}{3} + S_{3,2}(|x|)(x_1^2 - \tfrac{1}{3}|x|^2).$$

Unfortunately however, there is no simple expression for $S_{3,2}(r)$. The function $S_{3,2}(r)$ is given by

$$\begin{aligned} S_{3,2}(r) &= c_{3,2}F(2, -\tfrac{1}{2}; \tfrac{7}{2}; r^2) \\ &= \frac{\Gamma(\frac{3}{2})\Gamma(4)}{\Gamma(-\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(k+2)\Gamma(k-\frac{1}{2})}{\Gamma(k+\frac{7}{2})k!} r^{2k} \\ &= -\tfrac{3}{2} \sum_{k=0}^{\infty} \frac{(k+1)\Gamma(k-\frac{1}{2})}{\Gamma(k+\frac{7}{2})} r^{2k}. \end{aligned}$$

Thus

$$P[t_1^2](x) = \tfrac{1}{3} - \tfrac{3}{2}(x_1^2 - \tfrac{1}{3}|x|^2) \sum_{k=0}^{\infty} \frac{(k+1)\Gamma(k-\frac{1}{2})}{\Gamma(k+\frac{7}{2})} |x|^{2k}.$$

Using (5.27) we also have

$$S_{3,2}(r) = 4(1-r^2)^2 \sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{(2k+5)(2k+3)} r^{2k}.$$

Our next goal is to obtain an expansion of the Poisson kernel P in terms of the zonal harmonics. Fix a point $\eta \in S$. By considering the linear functional $\gamma : \mathcal{H}_m(S) \to \mathbb{R}$ defined by $\gamma(p) = p(\eta)$, it follows from the Riesz representation theorem that there exists a unique function $Z_\eta^{(m)} \in \mathcal{H}_m(S)$ such that

$$p(\eta) = \int_S p(t) Z_\eta^{(m)}(t)\, d\sigma(t)$$

for all $p \in \mathcal{H}_m(S)$. The spherical harmonic $Z_\eta^{(m)}$ is called the *zonal harmonic* of degree m with pole η. Set $Z_m(\eta,\zeta) = Z_\eta^{(m)}(\zeta)$. The zonal harmonic Z_m satisfies

(a) $Z_m(\eta,\zeta) = Z_m(\zeta,\eta)$,

(b) $Z_m(A\eta, A\zeta) = Z_m(\eta,\zeta)$ for all $A \in O(n)$, and

(c) $Z_m(\eta,\eta) = \|Z_\eta^{(m)}\|_2^2 = h_m = \dim \mathcal{H}_m(S)$.

Also, for fixed $\eta \in S$, the function $Z_\eta^{(m)}$ has a unique extension to a harmonic function on $\mathbb{R}^n$. This function will again be denoted by $Z_m(x,\eta)$. An explicit formula for Z_m is given in [2, Theorem 5.2.4]. Specifically

$$Z_m(x,\eta) = (n+2m-2)\sum_{k=0}^{[m/2]}(-1)^k\frac{n(n+2)\cdots(n+2m-2k-4)}{2^k k!(m-2k)!}\langle x,\eta\rangle^{m-2k}|x|^{2k}.$$

Our next step will be to prove the following expansion of the invariant Poisson kernel on B in terms of the zonal harmonics Z_m.

Theorem 5.28. *For $x \in B$, $t \in S$,*

$$P(x,t) = \sum_{\alpha=0}^{\infty}|x|^\alpha S_{n,\alpha}(|x|)Z_\alpha(\tfrac{x}{|x|},t) = \sum_{\alpha=0}^{\infty}S_{n,\alpha}(|x|)Z_\alpha(x,t), \tag{5.28}$$

where the series converges absolutely, and uniformly on compact subsets of B.

Proof. We first proof that the series (5.28) converges absolutely, and uniformly on compact subsets of B. Consider $S_{n,\alpha}(r)$ given by

$$S_{n,\alpha}(r) = c_{n,\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k(1-\frac12 n)_k}{(\alpha+\frac12 n)_k k!}\,r^{2k}.$$

Let $m = [n/2]$, and set

$$S_{n,\alpha}(r) = c_{n,\alpha}[P_m(r) + Q_m(r)],$$

where

$$P_m(r) = \sum_{k=0}^{m-1}\frac{(\alpha)_k(1-\frac12 n)_k}{(\alpha+\frac12 n)_k k!}\,r^{2k} \quad\text{and}\quad Q_m(r) = \sum_{k=m}^{\infty}\frac{(\alpha)_k(1-\frac12 n)_k}{(\alpha+\frac12 n)_k k!}\,r^{2k}.$$

If n is even, then $(1-\frac12 n)_k = 0$ for all $k \ge m$, and thus $Q_m(r) \equiv 0$. Now

$$|P_m(r)| \le \sum_{k=0}^{m-1}\frac{(\alpha)_k|(1-\frac12 n)_k|}{(\alpha+\frac12 n)_k k!}\,r^{2k}.$$

Since $(\alpha)_k/(\alpha+\frac{1}{2}n)_k \le 1$ for all k,

$$|P_m(r)| \le \sum_{k=0}^{m-1} \frac{|(1-\frac{1}{2}n)_k|}{k!} = C_n,$$

where C_n is a constant depending only on n.

Our next step is to obtain an estimate for $Q_m(r)$ when n is odd. For $k \ge m$ we have

$$(\gamma)_k = (\gamma)_m(\gamma+m)_{k-m}.$$

Thus

$$\begin{aligned} Q_m(r) &= \sum_{k=m} \frac{(\alpha)_k(1-\frac{1}{2}n)_k}{(\alpha+\frac{1}{2}n)_k k!}\, r^{2m} \\ &= \frac{(\alpha)_m(1-\frac{1}{2}n)_m}{(\alpha+\frac{1}{2}n)_m} r^{2m} \sum_{j=0}^{\infty} \frac{(\alpha+m)_j(1+m-\frac{1}{2}n)_j}{(\alpha+m+\frac{1}{2}n)_j(m+j)!}\, r^{2j}. \end{aligned}$$

As $(m+j)! \ge j!$ and $1+m-\frac{1}{2}n > 0$,

$$|Q_m(r)| \le \frac{|(1-\frac{1}{2}n)_m|\Gamma(\alpha+n)\Gamma(\alpha+\frac{1}{2}n)}{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2}n+m)}\, F(\alpha+m, 1+m-\tfrac{1}{2}n; \alpha+m+\tfrac{1}{2}n; r^2).$$

But $F(\alpha+m, 1+m-\frac{1}{2}n; \alpha+m+\frac{1}{2}n; r^2)$ is an increasing function of r. Thus by Identity (5.25)

$$F(\alpha+m, 1+m-\tfrac{1}{2}n; \alpha+m+\tfrac{1}{2}n; r^2) \le \frac{\Gamma(\alpha+m+\frac{1}{2}n)\Gamma(n-1-m)}{\Gamma(\frac{1}{2}n)\Gamma(\alpha+n-1)}.$$

Therefore

$$|Q_m(r)| \le \frac{|(1-\frac{1}{2}n)_m|\Gamma(n-1-m)}{\Gamma(\frac{1}{2}n)} \frac{\Gamma(\alpha+m)\Gamma(\alpha+\frac{1}{2}n)}{\Gamma(\alpha)\Gamma(\alpha+n-1)}.$$

But by (5.26)

$$c_{n,\alpha} = \frac{\Gamma(\frac{1}{2}n)\Gamma(\alpha+n-1)}{\Gamma(n-1)\Gamma(\alpha+\frac{1}{2}n)}.$$

Thus

$$|S_{n,\alpha}(r)| \le C_n' \frac{\Gamma(\alpha+n-1)}{\Gamma(\alpha+\frac{1}{2}n)} + D_n \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)},$$

where again C'_n and D_n are constants depending only on n. Using the fact that

$$\lim_{\alpha\to\infty} \alpha^{b-a}\frac{\Gamma(\alpha+a)}{\Gamma(\alpha+b)} = 1,$$

we have

$$\frac{\Gamma(\alpha+a)}{\Gamma(\alpha+b)} \approx \alpha^{a-b}.$$

Hence by the above

$$|S_{n,\alpha}(r)| \le C\,\alpha^{[n/2]},$$

where C is a constant depending only on n. Also, for all $\zeta, t \in S$,

$$|Z_\alpha(\zeta,t)| \le \|Z_\alpha\|_2^2 = h_\alpha,$$

where $h_\alpha = \dim(\mathcal{H}_\alpha(S))$. By [2, Chapter 5]

$$h_\alpha = \begin{pmatrix} n+\alpha-1 \\ n-1 \end{pmatrix} - \begin{pmatrix} n+\alpha-3 \\ n-1 \end{pmatrix}.$$

But then $h_\alpha \le C\alpha^{n-2}$. Hence

$$\sum_{\alpha=0}^{\infty} |S_{n,\alpha}(|x|)|\,|Z_\alpha(x,\zeta)| \le C\sum_{\alpha=0}^{\infty} |x|^\alpha \alpha^p,$$

where $p = n+[n/2]-2$. The series on the right however converges for all x, $|x|<1$, and uniformly for $0 \le |x| \le \rho$, whenever $0<\rho<1$ is fixed. This proves our assertion.

It only remains to be shown that the series converges to $P(x,t)$. By [2, Theorem 5.14], if $f \in L^2(S)$, then

$$f(\eta) = \sum_{\alpha=0}^{\infty} \langle f, Z_\eta^{(\alpha)}\rangle$$

in $L^2(S)$. In particular, for fixed $x \in B$,

$$P(x,t) = \sum_{\alpha}^{\infty} P[Z_t^{(\alpha)}](x),$$

which by Theorem 5.25

$$= \sum_{\alpha=0}^{\infty} S_{n,\alpha}(|x|) Z_\alpha(x,t),$$

from which the result now follows. $\square$

An immediate consequence of the previous theorem is the following.

Corollary 5.29. *If $f \in L^2(S)$,*

$$P[f](x) = \sum_{\alpha=0}^{\infty} S_{n,\alpha}(|x|) \int_S Z_\alpha(x,t) f(t)\, d\sigma(t),$$

where the series converges absolutely, and uniformly on compact subsets of B.

Proof. For $f \in L^2(S)$, $f(\eta) = \sum_{\alpha=0}^{\infty} \langle f, Z_\eta^{(\alpha)} \rangle$ in $L^2(S)$. Thus

$$\begin{aligned}\int_S P(x,\eta) f(\eta)\, d\sigma(\eta) &= \sum_{\alpha=0}^{\infty} \int_S P(x,\eta) \langle f, Z_\eta^{(\alpha)} \rangle\, d\sigma(\eta) \\ &= \sum_{\alpha=0}^{\infty} \int_S \int_S P(x,\eta) f(t) Z_\alpha(\eta,t)\, d\sigma(t) d\sigma(\eta)\end{aligned}$$

which by Fubini's theorem and Theorem 5.25

$$= \sum_{\alpha=0}^{\infty} S_{n,\alpha}(|x|) \int_S f(t) Z_\alpha(x,t)\, d\sigma(t). \quad \square$$

References

[**1**] P. Ahern, M. Flores, and W. Rudin, *An invariant volume mean value property*, J. Funct. Analysis **111** (1993), 380–397.

[**2**] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Springer – Verlag, New York, NY, 1992.

[**3**] A. F. Beardon, *The Geometry of Discrete Groups*, Springer–Verlag, New York, NY, 1983.

[**4**] I. Chavel, *Eigenvalues in Riemannian Geometry,*, vol. 115, Academic Press, Orlando, Florida, 1984.

[**5**] P. Duren, *Theory of H^p spaces*, Academic Press, New York, NY, 1970.

[**6**] G. B. Folland, *Spherical harmonic expansion of the Poisson Szegö kernel for the ball*, Proc. A. M. S. **47** (1975), 401 – 408.

[**7**] H. Furstenberg, *A Poisson formula for semisimple Lie groups*, Ann. of Math. **77** (1963), 335–386.

[**8**] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, London, England, 1959.

[**9**] S. Helgason, *Groups and Geometric Analysis*, Academic Press, Orlando, Florida, 1984.

[**10**] L. L. Helms, *Introduction to Potential Theory*, Wiley – Interscience, New York, NY, 1969.

[**11**] F. I. Karpelic, *The geometry of geodesics and the eigenfunctions of the Laplace operator on symmetric spaces*, Trans. Moscow Math. Soc. **14** (1965), 51–199.

[**12**] A. Koranyi, *The Poisson integral for generalized half–planes and bounded symmetric domains*, Ann. Math. **82** (1965), 332–350.

[**13**] ———, *Harmonic functions on Hermitian hyperbolic space*, Trans. Amer. Math. Soc. **135** (1969), 507–516.

[**14**] S. G. Krantz, *Function Theory of Several Complex Variable*, Wadsworth & Brooks/Cole, Pacific Grove, California, 1992.

[**15**] N. N. Lebedev, *Special Functions and their Applications*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965.

[**16**] M. Pavlovic, *Inequalities for the gradient of eigenfunctions of the invariant laplacian in the unit ball*, Indag. Mathem., N. S. **2** (1991), 89 – 98.

[**17**] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer–Verlag, New York, NY, 1980.

[**18**] I. Sokolnikoff, *Tensor Analysis*, Wiley, New York, N. Y., 1964.

[**19**] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.

[**20**] M. Stoll, *A characterization of Hardy spaces on the unit ball of $\mathbb{C}^n$*, J. London Math. Soc. **48** (1993), 126–136.

[**21**] ———, *Invariant Potential Theory in the Unit Ball of $\mathbb{C}^n$*, London Math. Soc. Lect. Notes **199** (1994).

[**22**] M. Tsuji, *Potential Theory*, Chelsea Publ. Co., New York, 1975.

[**23**] D. Ullrich, *Radial limits of $\mathcal{M}$–subharmonic functions*, Trans. Amer. Math. Soc. **292** (1985), 501–518.

[**24**] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, London, 1968.

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