Solutions to High School Math Contest

University of South Carolina, December 6, 2008

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1. **(c)** The integer is a multiple of 63. There are three such multiples between 600 and 800. They are 630, 693, and 756. Only 693 is odd.

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2. **(b)** Let x be the length of the man's shadow. Considering similar triangles we get $\dfrac{x}{x+5} = \dfrac{6}{16}$.

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3. **(a)** Since $x^4 + 5x^2 + 3 = (x^2+1)^2 + 3(x^2+1) - 1$, setting $y = x^2 + 1$ we get $P(y) = y^2 + 3y - 1$. Thus, $P(x^2 - 1) = (x^2 - 1)^2 + 3(x^2 - 1) - 1$.

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4. **(e)** Since $252 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$, one of the digits is 7, the integer must have three digits, and the remaining two digits are $6, 6$, or $4, 9$. There are five such odd integers: 667, 497, 947, 479, and 749.

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5. **(d)** If all statements are false, the day is either Monday, Saturday, or Sunday. If exactly one statement is true, the day is either Tuesday, Wednesday, or Friday. If exactly two statements are true, the day is Thursday. Only in the later case the number of correct statements uniquely determines the day of the week.

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6. **(a)** Inequalities ii and iii are true, $(a-b)^2 \geq 0$, so $a^2 + b^2 \geq 2ab \geq ab$. On the other hand, inequalities i and iv are false, for example take $a = 0.1$, $b = 0.1$.

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7. **(b)** We have $EF = \dfrac{2}{5}AC$. Thus, the ratio of the areas of $\triangle BEF$ and $\triangle BAC$ is $\dfrac{2}{5}$.

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8. **(d)** Consider an axis through the center of the clock with direction, the direction of the hands of the clock at 12:00. At 4:18 the angle between the axis and the hour hand is $\left(4 + \dfrac{18}{60}\right) \cdot \dfrac{360°}{12} = 129°$. The angle between the axis and the minute hand is $\dfrac{18}{60} \cdot 360° = 108°$.

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9. **(a)** Let G be the intersection point of lines $\overline{FA}$ and $\overline{CB}$; H - the intersection point of lines $\overline{BC}$ and $\overline{ED}$; and I - the intersection point of lines $\overline{DE}$ and $\overline{AF}$. Then $\triangle ABG$, $\triangle CDH$, and $\triangle EFI$ are equilateral with side length 1, and $\triangle GHI$ is equilateral with side length $2 + \sqrt{3}$. The area of an equilateral triangle with side length a is $\frac{\sqrt{3}a^2}{4}$. Thus, the area of the hexagon is

$\frac{\sqrt{3}}{4} \cdot \left((2+\sqrt{3})^2 - 3 \cdot 1^2\right)$.

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10. **(a)** There are $\binom{4}{2} = 6$ ways to select two white and two black balls, namely $BBWW$, $BWBW$, $BWWB$, $WBBW$, $WBWB$, and $WWBB$. The probability of each of the above 6 events is $p^2(1-p)^2$. (For example the probability of $WBBW$ is $(1-p) \cdot p \cdot p \cdot (1-p)$.)

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11. **(e)** The sum equals $(1-0.1) + (1-.01) + \cdots + (1-.0000000001) = 10 - .1111111111 = 9.8888888889$. Thus $A = 98888888889$.

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12. **(b)** We have $2^{2008} + 10^{2008} = 2^{2008}\left(1 + 5^{2008}\right)$. Now, 5 gives remainder 1 when divided by 4, and so does 5^{2008}. Thus 2 divides $1 + 5^{2008}$ but 4 does not.

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13. **(c)** The product we want to minimize equals $\dfrac{1+a+b+ab}{ab} = 1 + \dfrac{5}{ab}$. Thus, we need to maximize $ab = a(4-a) = 4a - a^2$. The maximum of a parabola that opens downwards is at its vertex, $a = 2$ in our case.

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14. **(e)** Since $\triangle QAB$ and $\triangle PDA$ are congruent, $\angle QAB = \angle PDA$. Also, $\triangle PCB$ and $\triangle QDC$ are congruent, so $\angle PCB = \angle QDC$. Thus, the sum of the measures of the three angles is $90°$.

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15. **(a)** Since $81 = 3^4$, $\log_3 81 = 4$, and $\log_{81} 3 = \frac{1}{4}$. So, the value of the expression is $\log_2\left(\frac{1}{4}\right)^4 = \log_2 2^{-8} = -8$.

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16. **(d)** Let M be the midpoint of the chord $\overline{AB}$, N - the midpoint of the chord $\overline{CD}$, and let O be the center of the circle. Then $AM = BM = 5$, and $CN = ND = 7$, so $NP = 5$. Moreover, $\overline{OM}$ is perpendicular to $\overline{AB}$, and $\overline{ON}$ is perpendicular to $\overline{CD}$. Thus, $PMON$ is a rectangle, and $NP = OM = 5$. Applying the Pythagorean Theorem to $\triangle OBM$ we get $OB = 5\sqrt{2}$.

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17. **(b)** We have $p(x) = 1 + x^2 + x^4 + x^6 + x^8 + x^{10} = \dfrac{x^{12}-1}{x^2-1} = \dfrac{x^6-1}{x^2-1} \cdot (x^6+1)$. Note that the polynomials $x^6 - 1$ and $\dfrac{x^6-1}{x^2-1} = 1 + x^2 + x^4$ have four common roots. (Also, $p(x)$ has no common roots with x^5+1, x^8+1, $x^{10}-1$, and $x^{12}+1$. For example, if we assume that r is a common root of $p(x)$ and x^5+1, we get $r^{12} = 1$, $r^5 = -1$, so $r^{10} = 1$, which implies $r^2 = 1$. On the other hand, if $r^2 = 1$, $p(r) = 6 \neq 0$ contradicting our assumption.)

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18. **(c)** Since both $\angle ACB$ and $\angle APB$ are right angles, the points P and C lie on the circle with diameter AB. Therefore $\angle PCB$ and $\angle PAB$ have the same measure.

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19. **(d)** Let $S = \dfrac{abc+1}{abc}$. The equations can be rewritten as $aS = \frac{1}{5}$, $bS = \frac{-1}{15}$, and $cS = \frac{1}{3}$. Since $S \neq 0$ we have $\dfrac{c-b}{c-a} = \dfrac{cS-bS}{cS-aS} = 3$. (Real numbers a, b, c which satisfy the above equation do exist. They are $a = 3T$, $b = -T$, $c = 5T$ where T is the unique real solution of the cubic equation $15x^3 - x^2 - 1 = 0$.)

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20. **(c)** Let H_1 be the point on line $\overline{AB}$ such that $\overline{PH_1}$ is perpendicular to $\overline{AB}$; let H_2 be the point on line $\overline{CD}$ such that $\overline{PH_2}$ is perpendicular to $\overline{CD}$; and let H_3 be the point on line $\overline{EF}$ such that $\overline{PH_3}$ is perpendicular to $\overline{EF}$. Then the sum of the areas of the shaded triangles equals $5(PH_1 + PH_2 + PH_3)$. Let G be the point where lines $\overline{AB}$ and $\overline{CD}$ intersect; H - the point where lines $\overline{CD}$ and $\overline{EF}$ intersect; and I - the point where lines $\overline{EF}$ and $\overline{AB}$ intersect. Clearly, $\triangle GHI$ is equilateral. Also, its area equals $15(PH_1 + PH_2 + PH_3)$. Therefore the sum $PH_1 + PH_2 + PH_3$ does not depend on the position of the point P, as long as P is inside $\triangle GHI$. If we move P to the center of the hexagon, $\triangle ABP$, $\triangle CDP$, and $\triangle EFP$ will be equilateral, and the sum of their areas will be $75\sqrt{3}$.

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21. **(d)** If x is a solution of the equation, then $lx = k\pi$ for some integers k and l with $1 \leq l \leq 12$. Thus $x = \frac{k}{l}\pi$. Since we want x to be in $(0, \pi]$, the number of solutions equals the number of fractions in $(0, 1]$ with numerator an integer, and denominator a positive integer ≤ 12. Since we want to avoid counting the same solution twice (for example $2\pi/3 = 4\pi/6$), we need to count only fractions in lowest terms. Let $\varphi(n)$ be the number of fractions in lowest terms in $(0, 1]$ with denominator n. Then $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(4) = 2$, $\varphi(5) = 4$, $\varphi(6) = 2$, $\varphi(7) = 6$, $\varphi(8) = 4$, $\varphi(9) = 6$, $\varphi(10) = 4$, $\varphi(11) = 10$, and $\varphi(12) = 4$. Adding the above values of φ we get that the number of solutions is 46.

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22. **(a)** Since $FB = FE$, $\angle FBE$ and $\angle FEB$ have the same measure. Similarly, $AE = AB$ implies that $\angle AEB$ and $\angle ABE$ have the same measure. Therefore $\angle ABF$ and $\angle AEF$ are both right angles. Moreover, since $EC = CF$, $\angle CEF$ has measure $45°$. Thus, $\angle DEA$ has measure $45°$ too, and $ED = AD = 21$. We get $AE = 21\sqrt{2}$, and $AB = AE = 21\sqrt{2}$.

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23. **(c)** After the first new animal arrives there will be either three groups of 2 animals or two groups of 3 animals. After the second new animal arrives, in both cases we will end up with two groups of 2 animals and one group of 3 animals. The third new animal will either $\mathcal{A}$: join a group of two animals, or $\mathcal{B}$: join a group of three animals. Event $\mathcal{A}$ happens with probability $2/3$ and we end up with one group of 2 animals and two groups of 3 animals; Event $\mathcal{B}$ happens with probability $1/3$ and we end up with four groups of 2 animals. If event $\mathcal{A}$ happens, the fourth new animal joins a group of 2 animals with probability $1/3$. If event $\mathcal{B}$ happens, the fourth animal joins a group of 2 animals with probability 1. The probability of the fourth new animal joining a group of 2 animals is $(2/3)(1/3) + (1/3)(1) = 5/9$.

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24. **(b)** Since DD_1A_1A is a rectangle, $DG = D_1G = A_1G = AG$. Similarly, $DF = C_1F = B_1F = AF$. Thus $\overline{GF}$ is parallel to $\overline{DC_1}$, so $\triangle EFG$ and $\triangle EC_1D$ are similar. Also, $\triangle EC_1D$ and $\triangle EAA_1$ are similar, which has several implications: (i) $DE = 2EA_1$ and $C_1E = 2EA$; (ii) the area of $\triangle EC_1D = 2/9$ (the triangle has base $2/3$ and height $2/3$); (iii) $4EG = ED$. Thus the area of $\triangle EFG$ is $1/16$th of the area of $\triangle EC_1D$.

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25. **(c)** We have $n^3 - 1 = (n^2 + n + 1)(n - 1)$ so $n^2 + n + 1$ divides $n^3 - 1$. Similarly, $n^3 - 1$ divides $n^{2010} - 1 = (n^3)^{670} - 1$. Thus, $n^2 + n + 1$ divides $n^{2010} - 1$. Therefore, $n^2 + n + 1$ divides $n^{2010} + 20$, if and only if, $n^2 + n + 1$ divides 21. Note that $n^2 + n + 1 > 0$ for all integers n. The only positive divisors of 21 are $1, 3, 7$, and 21. Solving the equations $n^2 + n + 1 = 1$, $n^2 + n + 1 = 3$, $n^2 + n + 1 = 7$, and $n^2 + n + 1 = 21$, we get the eight solutions $n = 0, -1, 1, -2, 2, -3, 4, -5$.

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26. **(d)** Denote the three sums by S_1, S_2, and S_3. We have $S_1 = 500 \cdot 1001$. Note that if a, b are integers and k is an odd positive integer, $a + b$ divides $a^k + b^k$. We can rewrite S_2 as $(1^{2007} + 1000^{2007}) + (2^{2007} + 999^{2007}) + \cdots + (500^{2007} + 501^{2007})$, so 1001 divides S_2. We get similarly that 1001 divides S_3. Another way to rewrite S_2 is $(1^{2007} + 999^{2007}) + (2^{2007} + 998^{2007}) + \cdots + (499^{2007} + 501^{2007}) + 500^{2007} + 1000^{2007}$, so 500 divides S_2. Similarly, 500 divides S_3. Therefore, the largest integer which divides all three sums is $S_1 = 500 \cdot 1001 = 500,500$.

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27. **(b)** First, note that $\triangle AFD$ and $\triangle AFE$ have the same area. Therefore, $\overline{ED}$ is parallel to $\overline{AF}$. Since $\triangle DEB$ and $\triangle ACB$ are similar, $BE/EC = 3/2$. Thus, the area of $\triangle AEB$ is $3/5$ths of the area of $\triangle ABC$.

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28. **(e)** Let $S = 1 + \frac{1}{2} + \cdots + \frac{1}{16}$. Then, $S = (1 + \frac{1}{16}) + (\frac{1}{2} + \frac{1}{15}) + \cdots + (\frac{1}{8} + \frac{1}{9}) = \frac{17}{1\cdot 16} + \frac{17}{2\cdot 15} + \cdots + \frac{17}{8\cdot 9}$. Therefore, 17 divides a. We claim that none of $2, 3, 5, 7, 11, 13$ divides a. We have $S - \frac{1}{13} = \frac{a_1}{b_1}$ where 13 does not divide b_1. Thus, $S = \frac{13a_1 + b_1}{13b_1}$. So, 13 does not divide a. Similarly, 11 does not divide a. Considering $S - \frac{1}{7} - \frac{1}{14} = S - \frac{3}{14}$, we get that 7 does not divide S. Similarly, analyzing $S - \frac{1}{5} - \frac{1}{10} - \frac{1}{15} = S - \frac{11}{30}$, we get that 5 does not divide S. Finally, considering $S - \frac{1}{9}$ and $S - \frac{1}{16}$, we conclude that 2 and 3 do not divide a. The smallest prime dividing a is 17. Indeed, $a = 2436559 = 17^2 \cdot 8431$ and $b = 720720 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

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29. **(c)** First, note that $\log_2 3 \cdot \log_3 4 = 2$. Let $f(x) = x + \dfrac{2}{x}$. We need to estimate $f(\log_2 3)$. The function f is increasing for $x > \sqrt{2}$. Indeed, if $x_1 > x_2 > \sqrt{2}$, $f(x_1) - f(x_2) = (x_1 - x_2)(1 - \dfrac{2}{x_1x_2}) > 0$. Next, $3^2 > 2^3$, so $3 > 2^{3/2}$, and $\log_2 3 > \log_2 2^{3/2} = 3/2$. Also, $2^8 > 3^5$, so $\log_2 3 < \log_2 2^{8/5} = 8/5$. Thus, $2.85 = f(8/5) > f(\log_2 3) > f(3/2) = 2.8\overline{3}$.

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30. **(d)** If a is an integer, then a^3 gives remainder 0, 1, or 6 when divided by 7 (just consider the remainders of 0^3, 1^3, 2^3, 3^3, 4^3, 5^3, and 6^3 when divided by 7). Then $a^3 + b^3$ gives remainder

0, 1, 2, 5, or 6 when divided by 7. Well, 7077283 gives remainder 3 when divided by 7. (The remaining numbers can be represented as a sum of two cubes of integers, $700056 = 40^3 + 86^3$, $707713 = 14^3 + 89^3$, $7000639 = 32^3 + 191^3$, and $7077915 = 3^3 + 192^3$.)