

**High School Math Contest**  
**University of South Carolina**  
**December 6, 2008**

1. An odd integer between 600 and 800 is divisible by 7 and also divisible by 9. What is the sum of its digits?

- (a) 7                      (b) 12                      (c) 18                      (d) 21                      (e) 27

2. At night a man who is 6 feet tall stands 5 feet away from a lamppost. The lamppost's lightbulb is 16 feet above the ground. How long is the man's shadow?

- (a)  $\frac{15}{8}$  feet                      (b) 3 feet                      (c)  $\frac{10}{3}$  feet                      (d) 4 feet                      (e) 5 feet

3. Given that  $P(x)$  is a polynomial such that  $P(x^2 + 1) = x^4 + 5x^2 + 3$ , what is  $P(x^2 - 1)$ ?

- (a)  $x^4 + x^2 - 3$                       (b)  $x^4 + 5x^2 - 1$                       (c)  $x^2(x + 1)(x - 1)$   
(d)  $x^4 + x^2 + 1$                       (e)  $x^4 - x^2 - 1$

4. How many positive odd integers  $n$  less than 1000 have the property that the product of the digits of  $n$  is 252?

- (a) 0                      (b) 1                      (c) 3                      (d) 4                      (e) 5

5. The following statements were made on the same day:

- It was Monday yesterday.
- Today is Thursday.
- The day after tomorrow will be Friday.
- Tomorrow will be Saturday.
- The day before yesterday was Tuesday.

Given that the number of correct statements above uniquely determines the day of the week the statements were made, what day of the week were the statements made?

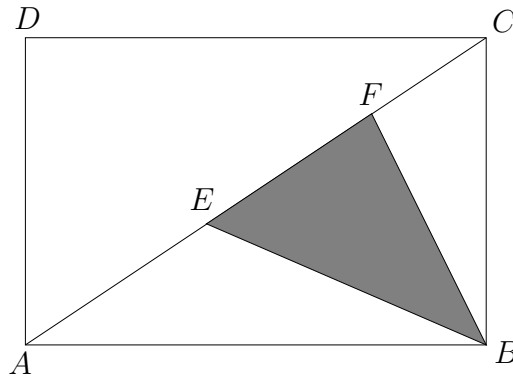
- (a) Monday   (b) Tuesday   (c) Wednesday   (d) Thursday   (e) none of the previous choices

6. Which of the following inequalities are true for all positive real numbers  $a$  and  $b$  ?

i.  $a^2 + b^2 \geq a + b$       ii.  $a^2 + b^2 \geq 2ab$       iii.  $a^2 + b^2 \geq ab$       iv.  $a^3 + b^3 \geq ab$

- (a) ii and iii only      (b) ii, iii and iv only      (c) ii only      (d) iv only      (e) i, ii and iii only

7. The area of the rectangle  $ABCD$  is 30. The points  $E$  and  $F$  are on the diagonal  $\overline{AC}$  as shown, and  $2(AE + FC) = 3EF$ . What is the area of  $\triangle BEF$  ?



- (a) 5      (b) 6      (c) 9      (d) 10      (e) cannot be determined

8. What is the measure of the acute angle between the hour and minute hands of a correctly working clock at 4:18 ?

- (a)  $12^\circ$       (b)  $15^\circ$       (c)  $18^\circ$       (d)  $21^\circ$       (e)  $24^\circ$

9. All interior angles of the hexagon  $ABCDEF$  have equal measure. Moreover,

$$AB = CD = EF = 1 \quad \text{and} \quad BC = DE = FA = \sqrt{3}.$$

What is the area of the hexagon?

- (a)  $\sqrt{3} + 3$       (b)  $3\sqrt{3} - 1$       (c) 4      (d)  $2\sqrt{3} + 1$       (e)  $3\sqrt{2}$

10. A box contains only white balls and black balls. Let  $p$  be the probability that a ball selected at random is black. Each time a ball is selected, it is placed back in the box before selecting the next ball. Four balls are selected at random. What is the probability that two of the four balls selected are black and two are white?

- (a)  $6p^2(1 - p)^2$       (b)  $4p^2(1 - p)^2$       (c)  $p^2(1 - p)^2$       (d)  $4p^2(1 - p^2)$       (e) 0.5

11. Let  $A$  be the positive integer satisfying the following equation.

$$\frac{9}{10} + \frac{99}{10^2} + \frac{999}{10^3} + \cdots + \frac{9999999999}{10^{10}} = \frac{A}{10^{10}}$$

How many 8's appear in the decimal representation of  $A$  ?

- (a) 0                      (b) 1                      (c) 5                      (d) 8                      (e) 9

12. What is the largest power of 2 that divides  $2^{2008} + 10^{2008}$  ?

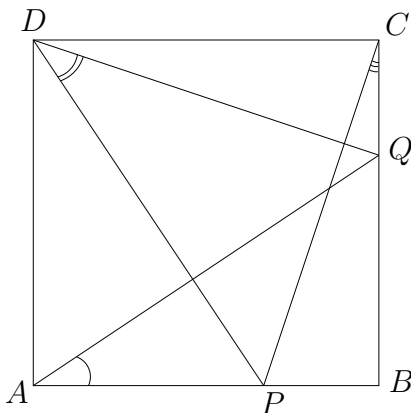
- (a)  $2^{2008}$               (b)  $2^{2009}$               (c)  $2^{2010}$               (d)  $2^{2011}$               (e)  $2^{2012}$

13. Given that  $a$  and  $b$  are positive real numbers with  $a + b = 4$ , what is the minimum value of

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) ?$$

- (a) 2                      (b)  $\frac{8}{3}$                       (c)  $\frac{9}{4}$                       (d) 3                      (e) 4

14. Let  $ABCD$  be a square. Point  $P$  is on side  $\overline{AB}$  with  $AP = 2BP$ . Point  $Q$  is on side  $\overline{BC}$  with  $BQ = 2CQ$ . What is the sum of the measures of the angles  $\angle QAB$ ,  $\angle PDQ$  and  $\angle PCB$  ?



- (a)  $60^\circ$               (b)  $75^\circ$               (c)  $100^\circ$               (d)  $120^\circ$               (e) none of the previous choices

15. What is the value of  $\log_2 \left( (\log_{81} 3)^{\log_3 81} \right)$  ?

- (a)  $-8$                       (b)  $-4 \log_2 3$               (c)  $-4$                       (d)  $-4 \log_3 2$               (e) 0

16. Let  $\overline{AB}$  and  $\overline{CD}$  be two chords of a circle that intersect at a point  $P$ . Suppose that  $AP = 4$ ,  $PB = 6$ ,  $CP = 2$ ,  $PD = 12$  and  $\angle APC = 90^\circ$ . What is the radius of the circle?

- (a)  $4\sqrt{3}$                       (b)  $3\sqrt{6}$                       (c) 8                      (d)  $5\sqrt{2}$                       (e)  $\sqrt{47}$

17. The polynomial

$$1 + x^2 + x^4 + x^6 + x^8 + x^{10}$$

has at least one root in common with which of the following?

- (a)  $x^5 + 1$                       (b)  $x^6 - 1$                       (c)  $x^8 + 1$                       (d)  $x^{10} - 1$                       (e)  $x^{12} + 1$

18. Let  $\triangle ABC$  be a right triangle with hypotenuse  $\overline{AB}$  and with the measure of  $\angle BAC$  equal to  $32^\circ$ . A square with side  $\overline{AB}$  is placed so that the interior of the square does not overlap the interior of  $\triangle ABC$ . Let  $P$  be the center of the square. What is the measure of  $\angle PCB$ ?

- (a)  $30^\circ$                       (b)  $32^\circ$                       (c)  $45^\circ$                       (d)  $58^\circ$                       (e)  $60^\circ$

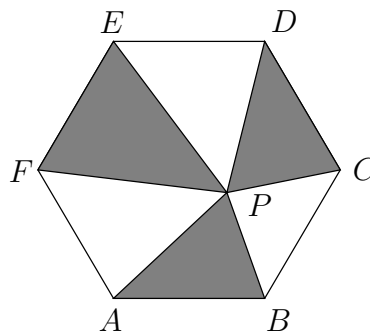
19. Let  $a$ ,  $b$  and  $c$  be real numbers which satisfy the three equations below.

$$a + \frac{1}{bc} = \frac{1}{5}, \quad b + \frac{1}{ac} = \frac{-1}{15}, \quad c + \frac{1}{ab} = \frac{1}{3}$$

What is the value of the quotient  $(c - b)/(c - a)$ ?

- (a)  $-5$                       (b)  $-3$                       (c) 1                      (d) 3                      (e) 5

20. Let  $ABCDEF$  be a regular hexagon with  $AB = 10$ . Let  $P$  be a point inside the hexagon with  $BP = 8$ . Denote by  $S$  the sum of the areas of the three triangles  $\triangle ABP$ ,  $\triangle CDP$  and  $\triangle EFP$ . What is the value of  $S$ ?



- (a)  $50\sqrt{3}$                       (b)  $75\sqrt{2}$                       (c)  $75\sqrt{3}$                       (d) 120                      (e) cannot be determined

21. How many solutions does the equation

$$\sin(x) \sin(2x) \sin(3x) \cdots \sin(11x) \sin(12x) = 0$$

have in the interval  $(0, \pi]$  ?

- (a) 11                      (b) 12                      (c) 24                      (d) 46                      (e) 68

22. The rectangle  $ABCD$  has the two sides  $\overline{AD}$  and  $\overline{BC}$  of length 21. Point  $F$  is on side  $\overline{BC}$ , and point  $E$  is on side  $\overline{CD}$ . Moreover,  $AB = AE$ ,  $CE = CF$  and  $FB = FE$ . What is  $AB$  ?

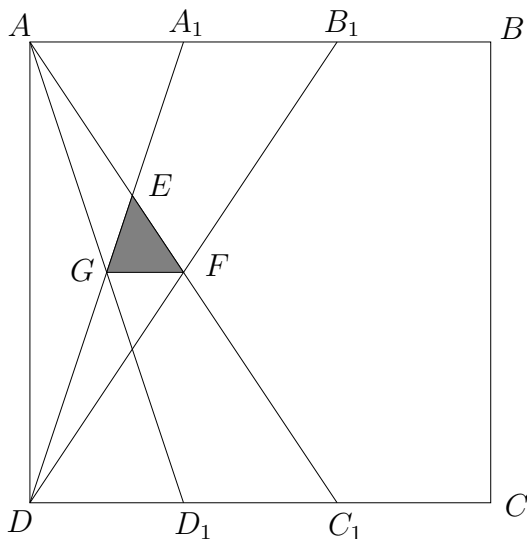
- (a)  $21\sqrt{2}$               (b) 28                      (c)  $7(\sqrt{2} + \sqrt{3})$               (d)  $4(3\sqrt{5} - 2)$               (e)  $21\sqrt{3}$

23. In a certain habitat, animals live in groups of two and groups of three. Each time a new animal arrives at the habitat, it randomly chooses a group. If the chosen group has two animals in it, the new animal joins the group forming a group of three animals. If the chosen group has three animals in it, the new animal takes one of the animals away from the group of three and forms a separate group of two animals.

If initially there are five animals in the habitat, and new animals arrive one at a time, what is the probability that the fourth new animal joins a group of two animals?

- (a)  $\frac{1}{3}$                       (b)  $\frac{1}{2}$                       (c)  $\frac{5}{9}$                       (d)  $\frac{5}{8}$                       (e) none of the previous choices

24. If  $ABCD$  is a square with  $AB = 1$  and  $AA_1 = A_1B_1 = B_1B = CC_1 = C_1D_1 = D_1D$  as shown below, what is the area of  $\triangle EFG$  ?



- (a)  $\frac{1}{60}$                       (b)  $\frac{1}{72}$                       (c)  $\frac{1}{84}$                       (d)  $\frac{1}{96}$                       (e) none of the previous choices

25. For how many integers  $n$  is  $n^2 + n + 1$  a divisor of  $n^{2010} + 20$  ?
- (a) 6                      (b) 7                      (c) 8                      (d) 10                      (e) 20

26. Let  $d$  be the largest positive integer that divides all three of the values of the sums below.

$$1 + 2 + 3 + \cdots + 999 + 1000$$

$$1^{2007} + 2^{2007} + 3^{2007} + \cdots + 999^{2007} + 1000^{2007}$$

$$1^{2009} + 2^{2009} + 3^{2009} + \cdots + 999^{2009} + 1000^{2009}$$

Which of the following is true about  $d$  ?

- (a)  $d = 1$       (b)  $1 < d \leq 10^2$       (c)  $10^2 < d \leq 10^4$       (d)  $10^4 < d \leq 10^6$       (e)  $d > 10^6$
27. Let  $A$ ,  $B$  and  $C$  denote the vertices of a triangle with area 10. Let point  $D$  be on side  $\overline{AB}$ , point  $E$  be on side  $\overline{BC}$  and point  $F$  be on side  $\overline{CA}$  with  $AD = 2$  and  $DB = 3$ . The area of  $\triangle ABE$  and the area of quadrilateral  $DBEF$  are the same. What is the value of this area?
- (a) 5.5                      (b) 6                      (c) 7                      (d) 8                      (e) 8.25

28. Let  $a$  and  $b$  be positive integers with no common prime factor satisfying the equation below.

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{15} + \frac{1}{16} = \frac{a}{b}$$

What is the smallest prime divisor of  $a$  ?

- (a) 3                      (b) 7                      (c) 13                      (d) 19                      (e) none of the previous choices
29. Which of the following numbers is the closest in value to  $\log_2 3 + \log_3 4$  ?
- (a) 2.75                      (b) 2.8                      (c) 2.85                      (d) 2.9                      (e)  $\pi$

30. Exactly one of the following numbers cannot be written as  $a^3 + b^3$  where  $a$  and  $b$  are integers. Which number is it?
- (a) 700056                      (b) 707713                      (c) 7000639                      (d) 7077283                      (e) 7077915