High School Math Contest
University of South Carolina
December 8, 2007

1. Jack and Jill collect ladybugs. Jack only collects the ones with 2 spots, and Jill only collects the ones with 7 spots. Jack has 5 more ladybugs than Jill. The total number of spots found on all of their ladybugs is 100. How many ladybugs do they have in their combined collection?

   (a) 17 (b) 21 (c) 23 (d) 25 (e) 35

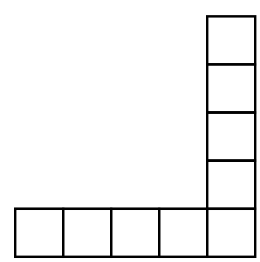
2. Jerry had an average score of 85 on his first eight quizzes. He had an average score of 81 on his first nine quizzes. What score did he receive on his ninth quiz?

   (a) 49 (b) 51 (c) 53 (d) 55 (e) 57

3. A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these lines?

   (a) 4 (b) 6 (c) 8 (d) 10 (e) 12

4. Each of the 9 squares shown is to contain one number chosen from 1, 2, 3, 4, 5, 6, 7, 8, and 9. No number is to be repeated. Suppose the sum of the 5 squares aligned vertically on the right is 32 and that the sum of the 5 squares aligned horizontally on the bottom is 20. What number goes in the shared corner square?



   (a) 3 (b) 4 (c) 5 (d) 6 (e) 7

5. Which of the numbers 1, 2, 3, 4, or 5 is nearest in value to the sum

$$\frac{2007}{2999} + \frac{8001}{5998} + \frac{2001}{3999}?$$

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

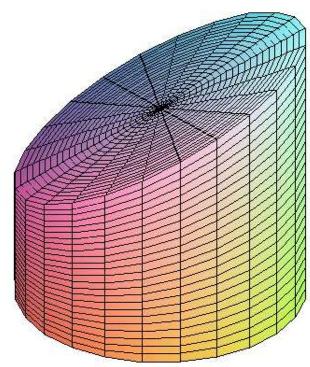
6. Given that $29a031 \times 342 = 100900b02$ where a and b denote two missing digits, what is the value of $a + b$?

(a) 7 (b) 8 (c) 9 (d) 10 (e) 11

7. Exactly one tenth of the students at Dave's high school are male. Exactly one third of all seniors at his school are male. Let p be the probability that a male at his school is a senior and let q be the probability that a female at his school is a senior. What is the value of $\dfrac{p}{q}$?

(a) 4 (b) 4.25 (c) 4.5 (d) 4.75 (e) 5

8. A cylinder is sliced by a plane to form the solid shown. The base edge of the solid is a circle of radius 3. The top edge is an ellipse. The highest point on the ellipse is 6 units above the base. The lowest point on the ellipse is 2 units above the base. What is the volume, in cubic units, of the solid?



(a) 24π (b) 30π (c) 36π (d) 42π (e) 48π

9. Suppose that $f(x) = x^x$ and $g(x) = x^{2x}$. Which of the functions below is equal to $f(g(x))$?

(a) x^{3x} (b) $x^{x^{2x}}$ (c) $x^{2x^{2x}}$ (d) $x^{2x^{3x}}$ (e) $x^{2x^{2x+1}}$

10. The king took a cup filled with water and drank 1/5 of its contents. When the king looked away, the court jester refilled the cup by adding alcohol to the remaining water and then stirred. The king drank 1/4 of this liquid mixture. When the king looked away again, the court jester refilled the cup with more alcohol and stirred. The king drank 1/3 of this liquid mixture. When the king looked away a third time, the court jester refilled the cup with more alcohol. What percent of this final mixture was alcohol?

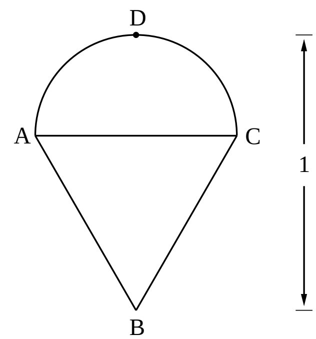
(a) 50% (b) 60% (c) 70% (d) 75% (e) 80%

11. What is the value of the following product?

$$\sin\frac{\pi}{32}\cos\frac{\pi}{32}\cos\frac{\pi}{16}\cos\frac{\pi}{8}\cos\frac{\pi}{4}$$

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{1}{16}$ (e) $\frac{1}{32}$

12. A semicircle is placed upon one side of equilateral $\triangle ABC$ as shown. The point halfway along the arc of the semicircle is labeled D. Given that $BD = 1$, what is the length of each side in the equilateral triangle?



(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3-\sqrt{3}}{2}$ (e) $\sqrt{3}-1$

13. Let $P_n = 1^n + 2^n + 3^n + 4^n$. Find the number of integers n for which $1 \le n \le 100$ and P_n is a multiple of 5.

(a) 68 (b) 75 (c) 86 (d) 98 (e) 100

14. Let a, b, c, d be positive real numbers with $a < b < c < d$. Given that a, b, c, d are the first four terms in an arithmetic sequence, and a, b, d are the first three terms in a geometric sequence, what is the value of $\frac{ad}{bc}$?

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{4}{5}$ (e) 1

15. Two points A and B lie on a sphere of radius 12. The length of the straight line segment joining A and B is $12\sqrt{3}$. What is the length of the shortest path from A to B if every point of the path is on the sphere of radius 12 ?

(a) 6π (b) 8π (c) 9π (d) 10π (e) 12π

16. Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$. What is the maximum value of $4x - 9y$?

(a) 15 (b) $9\sqrt{3}$ (c) 16 (d) $12\sqrt{2}$ (e) 18

17. What is the number of 7-element subsets of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for which the sum of those 7 elements is a multiple of 3 ?

(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

18. How many triples (a, b, c) of real numbers satisfy the equations

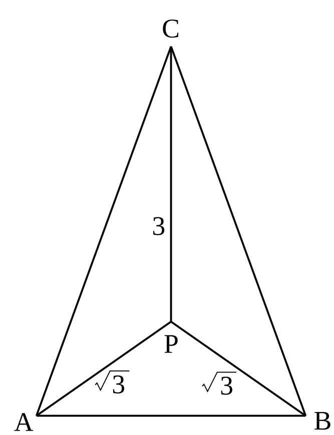
$$ab = c,\ ac = b,\ \text{and}\ \ bc = a\,?$$

(a) 2 (b) 4 (c) 5 (d) 6 (e) 8

19. Let x and y be real numbers with $x + y = 1$ and $\left(x^2 + y^2\right)\left(x^3 + y^3\right) = 12$. What is the value of $x^2 + y^2$?

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 2 (d) 3 (e) 4

20. In the figure shown, line $\overleftrightarrow{AP}$ bisects $\angle CAB$, line $\overleftrightarrow{BP}$ bisects $\angle ABC$, $AP = \sqrt{3}$, $BP = \sqrt{3}$, and $CP = 3$. What is the area of $\triangle ABC$?



(a) 4 (b) $3\sqrt{3}$ (c) $4\sqrt{2}$ (d) 6 (e) $4\sqrt{3}$

21. What is the length of the shortest path that begins at the point $(2, 5)$, touches the x-axis and then ends at a point on the circle

$$(x + 6)^2 + (y - 10)^2 = 16\,?$$

(a) 12 (b) 13 (c) $4\sqrt{10}$ (d) $6\sqrt{5}$ (e) $4 + \sqrt{89}$

22. For how many prime pairs (p, q) does there exist an integer n such that

$$\left(p^2 + 1\right)\left(q^2 + 1\right) = n^2 + 1\,?$$

(a) 2 (b) 4 (c) 6 (d) 8 (e) infinitely many

23. If 135^k divides $2007!$ and 135^{k+1} does not, then what is the value of k ?

(a) 250 (b) 333 (c) 500 (d) 666 (e) 1000

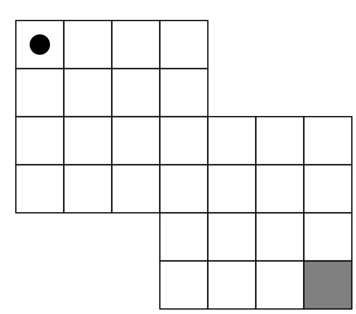
24. Let $a_1, a_2, \ldots, a_r$ be r not necessarily distinct positive integers such that

$$(x + a_1)(x + a_2)\cdots(x + a_r) = x^r + 230x^{r-1} + \cdots + 2007,$$

where each missing term on the right has degree between 1 and $r - 2$ inclusive. What is the value of r ?

(a) 4 (b) 12 (c) 63 (d) 669 (e) 2007

25. A game piece is placed on the upper left square of a game board with 30 squares as shown. By a sequence of 11 moves, each from a given square to an adjoining square either to the right or down, the piece is to go from the upper left square to the bottom right shaded square. How many 11-move sequences are possible?



(a) 300 (b) 330 (c) 360 (d) 400 (e) 462

26. Let S be the set $\{1, 2, 3, \ldots, n\}$ consisting of the first n positive integers. What is the maximum value of n for which every 100-element subset of S contains two integers which differ by 25 ?

(a) 171 (b) 172 (c) 173 (d) 174 (e) 175

27. What is the value of the following sum?

$$\frac{3}{1! + 2! + 3!} + \frac{4}{2! + 3! + 4!} + \frac{5}{3! + 4! + 5!} + \cdots + \frac{2007}{2005! + 2006! + 2007!}$$

(a) $\frac{2007! + 2}{2 \cdot 2007!}$ (b) $\frac{2007! + 1}{2 \cdot 2007!}$ (c) $\frac{2007! - 1}{2 \cdot 2007!}$

(d) $\frac{2007! - 2}{2 \cdot 2007!}$ (e) $\frac{2007! - 3}{2 \cdot 2007!}$

28. Katie has a collection of red marbles and blue marbles. The number of red marbles is either 7, 9, 11, 13, or 15. If two marbles are chosen simultaneously and at random from her collection, then the probability they have different colors is 0.5. How many red marbles are in Katie's collection?

(a) 7 (b) 9 (c) 11 (d) 13 (e) 15

29. What is the smallest positive integer m such that the following equation holds for some polynomials $u(x)$ and $v(x)$ with integer coefficients?

$$(x + 2)(x + 5)(x + 7)\, u(x) - (x - 2)(x - 5)(x - 7)\, v(x) = m$$

(a) 70 (b) 1260 (c) 1680 (d) 3780 (e) 7560

30. A positive integer n is called a "good" number if $n^3 + 7n - 133 = m^3$ for some positive integer m. What is the sum of all "good" numbers?

(a) 24 (b) 25 (c) 29 (d) 30 (e) ∞