

High School Math Contest
University of South Carolina
January 21, 2006

1. What is the value of the product $\left(2 + \frac{1}{2} + \frac{3}{4}\right) \cdot \left(1 + \frac{1}{3}\right)$?

- (a) $\frac{19}{3}$ (b) $\frac{17}{3}$ (c) $\frac{16}{3}$ (d) $\frac{14}{3}$ (e) $\frac{13}{3}$

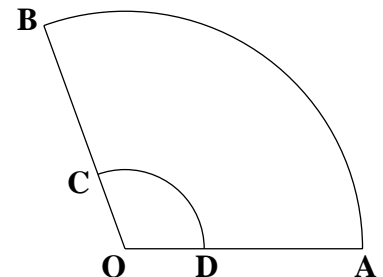
2. A woman has \$2.15 in change in her purse, comprised entirely of dimes and quarters. Given that there are more quarters than dimes in her purse, what is the total number of coins?

- (a) 9 (b) 10 (c) 11 (d) 12 (e) 13

3. If $\frac{28x}{45} = \frac{77}{165}$, then what is the value of x ?

- (a) 0.625 (b) 0.75 (c) 0.8 (d) $0.\bar{8}$ (e) 0.925

4. In the figure shown, OAB and ODC are sectors of two concentric circles centered at O . The length of \overline{OD} is one third the length of \overline{OA} . What is the ratio of the area of the region $ABCD$ to the area of the sector ODC ?



- (a) 7 (b) 8 (c) 10 (d) 11 (e) 12

5. A group containing boys and girls took a test. If exactly $\frac{2}{3}$ of the boys and exactly $\frac{3}{4}$ of the girls passed the test, and if an equal number of boys and girls passed the test, then what fraction of the entire group passed the test?

- (a) $\frac{11}{16}$ (b) $\frac{12}{17}$ (c) $\frac{13}{18}$ (d) $\frac{14}{19}$ (e) $\frac{17}{23}$

6. A digit is placed in each empty square in the grid to the right so that

- each row contains each of the digits 1, 2, 3, 4, 5.
- each column contains each of the digits 1, 2, 3, 4, 5.

	5	4		
1	3			
		5	3	
2		3	1	

What digit is placed in the square at the bottom right corner of the grid?

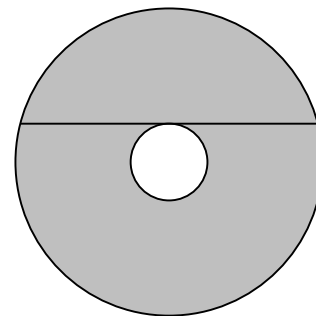
- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5
7. What is the last digit d of the 9-digit number $19700019d$, given that the number is prime?
- (a) 1 (b) 3 (c) 5 (d) 7 (e) 9
8. The equilateral $\triangle ABC$ has side length 4, and D is a point on the line segment \overline{BC} . Denote by S_1 the area of $\triangle ABD$, and by S_2 the area of $\triangle ACD$. What is the largest possible value of the product $S_1 \cdot S_2$?
- (a) 8 (b) 9 (c) 10 (d) 12 (e) 16

9. What is the value of $\frac{\log_2 3 \cdot \log_4 5 \cdot \log_6 7}{\log_4 3 \cdot \log_6 5 \cdot \log_8 7}$?

- (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 2 (d) 3 (e) 4

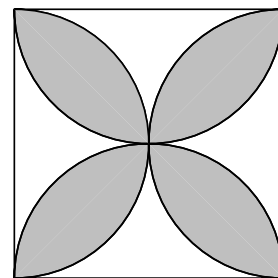
10. What is the remainder when $x^{2006} - x^{2005} + (x + 1)^2$ is divided by $x^2 - 1$?
- (a) 2 (b) 4 (c) $2x$ (d) $2x + 2$ (e) $x + 3$

11. The circles in the figure shown are concentric. The chord shown is tangent to the inner circle and has length 12. What is the area of the shaded region?



- (a) 24π (b) 32π (c) 36π (d) 40π (e) 48π
12. If the number 86 in base 10 is represented as 321 in base b , then 123 in base b can be represented as which of the following numbers in base 10 ?
- (a) 27 (b) 38 (c) 51 (d) 66 (e) 83
13. How many 5-digit numbers with all digits non-zero and no digit repeated are divisible by 25 ?
- (a) 360 (b) 420 (c) 450 (d) 480 (e) 500
14. Suppose that $f(x)$ is a function such that $3f(x) + 2f(1 - x) = 2x + 9$ for every real number x . What is the value of $f(2)$?
- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5
15. A right triangle with integer side lengths a , b , and c satisfies $a < b < c$ and $a + c = 49$. What is the area of the right triangle?
- (a) 105 (b) 140 (c) 175 (d) 210 (e) 245
16. How many rational numbers are solutions to $x^4 + 8x^3 - 40x + 125 = 0$?
- (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

17. In the figure shown, four semicircles are drawn inside a square with side length 1, and a portion of the drawing is shaded in. Which of the following numbers is closest in value to the area of the shaded portion?



- (a) 0.3 (b) 0.4 (c) 0.5 (d) 0.6 (e) 0.7

18. There are positive integers a , b , and c which satisfy the system of two equations below.

$$\begin{cases} c^2 - a^2 - b^2 = 101 \\ ab = 72 \end{cases}$$

What is the value of $a + b + c$?

- (a) 43 (b) 45 (c) 47 (d) 51 (e) 57

19. Each of two boxes contains 20 marbles, and each marble is either black or white. The total number of black marbles is different from the total number of white marbles. One marble is drawn at random from each box. The probability that both marbles are white is 0.21. What is the probability that both are black?

- (a) 0.22 (b) 0.23 (c) 0.24 (d) 0.25 (e) 0.26

20. How many positive real numbers x are solutions to the equation below?

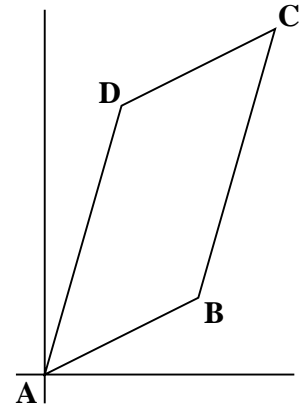
$$\sqrt{x} = |x^4 - 1|$$

- (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

21. Let a be a real number such that the polynomials $x^3 + (a - 1)x^2 - ax + 1$ and $x^2 + ax + 1$ have a common root. What is the value of a ?

- (a) $-\frac{5}{2}$ (b) -2 (c) 1 (d) $\frac{5}{3}$ (e) 3

22. In the figure shown, $ABCD$ is a parallelogram with $A = (0, 0)$, $B = (20, 10)$, and $D = (10, y)$. If the area of the parallelogram is 600, then what is the value of y ?



- (a) 32 (b) 33 (c) 34 (d) 35 (e) 36
23. How many positive integers n have a remainder of 6 when 2006 is divided by n ?
- (a) 8 (b) 10 (c) 12 (d) 16 (e) 18
24. Suppose the roots of the quadratic equation $x^2 + ax + b = 0$ are $\sin 15^\circ$ and $\cos 15^\circ$. What is the value of $a^4 - b^2$?
- (a) -1 (b) 1 (c) $\frac{35}{16}$ (d) $1 + \sqrt{2}$ (e) $3\sqrt{2} - 1$

25. Define a sequence by $s_1 = 2^{2006}$ and for $n \geq 1$,

$$s_{n+1} = \begin{cases} \log_2 s_n & \text{if } s_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the smallest integer n such that $s_n < 1$.

- (a) 2 (b) 3 (c) 4 (d) 5 (e) 6

26. A spy had to send the 4-digit code \boxed{abcd} to headquarters. For security reasons, he sent instead the 9 separate 4-digit codes shown. In each of the 9 codes, at least one of the digits a , b , c , and d occurs in its correct position. What is the value of d ?

2186
4351
4521
5127
5916
6384
6924
8253
8517

- (a) 1 (b) 3 (c) 4 (d) 6 (e) 7
27. Suppose that the complex numbers $1 + i$ and $1 + 2i$ are both solutions to the equation $x^5 - 6x^4 + Ax^3 + Bx^2 + Cx + D = 0$ where A , B , C , and D are integers. What is the value of $A + B + C + D$?
- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5
28. How many real numbers x satisfy the inequality below?

$$|x^4 - 4x^2 - 6| \geq |x^4 - 4x^2 + 14|$$

- (a) 0 (b) 1 (c) 2 (d) 4 (e) infinitely many
29. For θ in the domain of the function

$$f(\theta) = \cos \theta + \frac{\cos(3\theta)}{\cos \theta},$$

what is the least value of $f(\theta)$?

- (a) -2 (b) -3 (c) $-49/16$ (d) $-28/9$ (e) -4
30. For each positive integer m , let $s(m)$ denote the sum of the digits of m . What is the value of

$$s(s(s(2^{2006}))) ?$$

- (a) 2 (b) 4 (c) 8 (d) 22 (e) 40