High School Math Contest
University of South Carolina
December 9, 2006

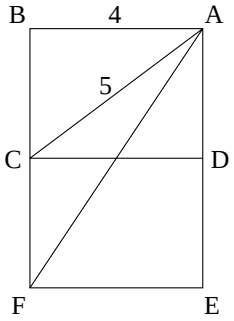
1. What are the last two digits of 5^{2007} ?

   (a) 75 (b) 65 (c) 55 (d) 25 (e) 05

2. The length of the shorter side of a rectangle is 2 units. The length of each diagonal is 4 units. What is the acute angle between the diagonals?

   (a) 15° (b) 22.5° (c) 45° (d) 60° (e) 75°

3. In the figure shown, $ABCD$ and $DCFE$ are rectangles with $AB = 4$, $AC = 5$, and $BC = CF$. What is AF ?



   (a) $5\sqrt{2}$ (b) $4\sqrt{3}$ (c) $3\sqrt{5}$ (d) $3\sqrt{6}$ (e) $2\sqrt{13}$

4. If $x + y = 0$ and $x \neq 0$, then what is the value of $\dfrac{x^{2007}}{y^{2007}}$?

   (a) -2007 (b) -1 (c) 0 (d) 1 (e) 2007

5. Two perpendicular line segments divide a large rectangle into 4 small rectangles. The areas of 3 of these 4 small rectangles are shown. What is the area of the other small rectangle?

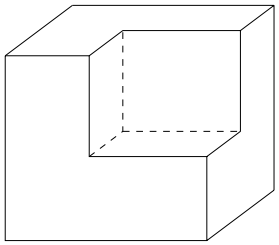
6	9
8	

   (a) 12 (b) 13 (c) 14 (d) 15 (e) 16

6. Joe has a total of \$200 in his two pockets. He takes one fourth of the money in his left pocket and puts it in his right pocket. He then takes \$20 from his left pocket and puts it in his right pocket. If he now has an equal amount of money in each pocket, then how much money did he originally have in his left pocket?

(a) \$120 (b) \$130 (c) \$140 (d) \$150 (e) \$160

7. The volume of a large cube is 125 cubic inches. A new shape is formed by removing a smaller cube from one corner of the large cube. The surface area of this new shape in square inches is

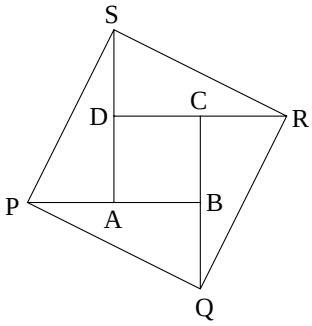


(a) 120 (b) 150 (c) 180 (d) 225 (e) 250

8. In a class of 100 students, there are 50 who play soccer, 45 who play basketball, and 50 who play volleyball. Only 15 of these students play all three sports. Everyone plays at least one of these sports. How many of the students play exactly two of these sports?

(a) 15 (b) 20 (c) 25 (d) 30 (e) 35

9. Suppose $ABCD$ is a square, and that A, B, C, and D are the midpoints of $\overline{BP}$, $\overline{CQ}$, $\overline{DR}$, and $\overline{AS}$, respectively. What is the ratio of the area of $PQRS$ to the area of $ABCD$?



(a) 4 (b) $3\sqrt{2}$ (c) 5 (d) $3\sqrt{3}$ (e) 6

10. Fresh grapes contain 80% water by weight, whereas dried grapes contain 15% water by weight. How many pounds of dried grapes can be obtained from 34 pounds of fresh grapes?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

11. For which value of θ listed below is it true that

$$2^{\sin\theta} > 1 \quad \text{and} \quad 3^{\cos\theta} < 1?$$

(a) 70° (b) 140° (c) 210° (d) 280° (e) 350°

12. The sum of seven consecutive integers is 980. How many of them are prime?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

13. It rained on exactly 7 of the days during Jane's summer holiday trip. On each day that it rained, it rained either in the morning or the afternoon but not both. There were exactly 5 afternoons when it did not rain and exactly 6 mornings when it did not rain. How many days did the trip last?

(a) 7 (b) 8 (c) 9 (d) 10 (e) 11

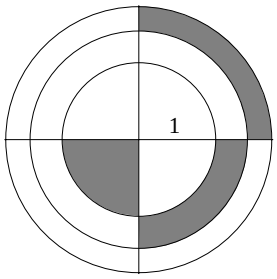
14. Define a sequence by $b_1 = 2$ and

$$b_{n+1} = \frac{1 + b_n}{1 - b_n} \quad \text{for } n \geq 1.$$

What is the value of b_{2006} ?

(a) -2007 (b) -3 (c) $-1/2$ (d) 2 (e) 3

15. In the figure shown, two perpendicular lines intersect at the center of three concentric circles. Each shaded region in the figure has the same area. If the smallest circle has radius 1, then what is the product of the 3 radii?



(a) $\sqrt{6}$ (b) 2.5 (c) $\dfrac{3\sqrt{3}}{2}$ (d) $2\sqrt{2}$ (e) π

16. What is the value of the following product?

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{49^2}\right)\left(1-\frac{1}{50^2}\right)$$

(a) $\frac{23}{50}$ (b) $\frac{27}{50}$ (c) $\frac{25}{51}$ (d) $\frac{26}{51}$ (e) $\frac{51}{100}$

17. Suppose that 25% of all the wise people are nice and half of all the nice people are wise. Suppose further that 25% of all the people are neither wise nor nice. What percent of all the people are both wise and nice?

(a) 10% (b) 15% (c) 20% (d) 25% (e) 30%

18. If $x^2+x+1=0$, then what is the value of $\left(x^3+\frac{1}{x^3}\right)^3$?

(a) -8 (b) -1 (c) 0 (d) 1 (e) 8

19. Find the smallest positive integer n such that 11^n-1 is divisible by 105.

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

20. How many polynomials are there of the form x^3-8x^2+cx+d such that c and d are real numbers and the three roots of the polynomial are distinct positive integers?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5

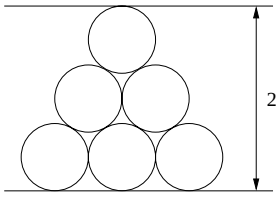
21. A man caught some fish. The 2 heaviest fish had a combined weight which was 25% of the total weight of all the fish. The 5 lightest fish had a combined weight which was 45% of the total weight of all the fish. He put the 2 heaviest fish in the freezer and ate the 5 lightest fish for lunch. His cat took all the remaining fish. How many fish did his cat take?

(a) 8 (b) 6 (c) 4 (d) 3 (e) 2

22. Suppose that 10 teams participated in a soccer tournament where each team played exactly one game with each of the other teams. The winner of each game received 3 points, while the loser received 0 points. In case of a tie, both teams received 1 point. At the end of the tournament, the 10 teams received a total of 130 points. How many games ended in a tie?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

23. The triangle-like shape in the diagram is formed from 6 identical close-packed circles. If the height of the shape is 2, then what is the radius of each circle?



(a) $\frac{1}{1+\sqrt{3}}$ (b) $\frac{2}{1+\sqrt{3}}$ (c) $\frac{1}{2+\sqrt{3}}$ (d) $\frac{2}{2+\sqrt{3}}$ (e) $\frac{1}{3}$

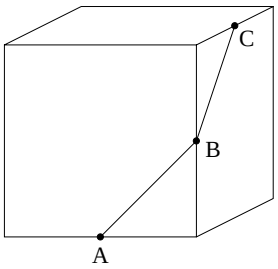
24. Let $f(x)$ be a function such that $f(x) + f\left(\frac{1}{1-x}\right) = x$ for all x not equal to 0 or 1. What is the value of $f(2)$?

(a) $\frac{1}{4}$ (b) $\frac{3}{4}$ (c) $\frac{5}{4}$ (d) $\frac{7}{4}$ (e) $\frac{9}{4}$

25. Jerry wrote down all the positive integers that have at most 7 digits and contain only the digits 0 and 1. How many times did he write down the digit 1 ?

(a) 128 (b) 224 (c) 288 (d) 448 (e) 512

26. The midpoints of three of the edges of a cube are labelled A, B, and C as shown in the diagram. What is the measure of $\angle ABC$?



(a) 90° (b) 105° (c) 120° (d) 135° (e) 150°

27. On Pluto, the inhabitants use the same mathematical operators that we do (+, −, etc.), and they also use an operator @ that we do not know. Scientists have determined that the following are true for any real numbers x and y:

$$x@0 = x$$

$$x@y = y@x$$

$$(x+1)@y = (x@y) + y + 1$$

What is the value of 12@5 ?

(a) 53 (b) 59 (c) 65 (d) 71 (e) 77

28. How many real numbers x satisfy the following equation?

$$\sqrt{3+\sqrt{3+\sqrt{3+x}}} = x$$

(a) 0 (b) 1 (c) 3 (d) 4 (e) 8

29. Let $f(x)$ be a polynomial of degree 2006 satisfying

$$f(k) = \frac{1}{k} \quad \text{for} \quad 1 \leq k \leq 2007.$$

What is the value of $f(2008)$?

(a) 0 (b) $\frac{1}{2008}$ (c) $\frac{2}{2008}$ (d) $\frac{3}{2008}$ (e) $\frac{4}{2008}$

30. USC invited each South Carolina high school to send up to 39 students to watch a football game. A section which has 199 seats in each row is reserved for those students. What is the least number of rows needed to guarantee that if 2006 students show up, then all students from the same high school can be seated in the same row?

(a) 11 (b) 12 (c) 13 (d) 14 (e) 15